Sec 8.4 Hyperplanes

Def A line in $\mathbb{R}^n$ is a flat of dimension 1

A hyperplane in $\mathbb{R}^n$ is a flat of dimension $n-1$. } Recall from Sec 8.1

Ex A hyperplane in $\mathbb{R}^2$ is a line.

An implicit equation of a line in $\mathbb{R}^2$ has the form $\underbrace{ax+by}_{\text{a linear expression}}=d$

Ex A hyperplane in $\mathbb{R}^3$ is a plane.

An implicit equation of a plane in $\mathbb{R}^3$ has the form $\underbrace{ax+by+cz}_{\text{a linear expression}}=d$.

So a plane is the set of all points at which a linear expression has a fixed value d.

Def • A linear functional on $\mathbb{R}^n$ is a linear map $f: \mathbb{R}^n \to \mathbb{R}$.

• If d is a scalar in $\mathbb{R}$, then the symbol

$$[f: d]$$

denotes the set of all $\vec{x}$ in $\mathbb{R}^n$ at which the value of f is d.

That is,

$$[f:d] = \{\vec{x} \in \mathbb{R}^n : f(\vec{x}) = d\}.$$

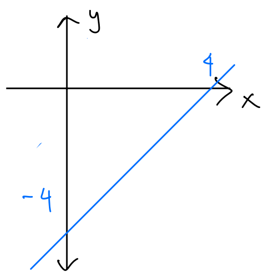
• The zero functional is the map $f(\vec{x})=0$ for all $\vec{x}$ in $\mathbb{R}^n$.

All other linear functionals on $\mathbb{R}^n$ are called nonzero.

Ex The line $x-y=4$ in $\mathbb{R}^2$ is a hyperplane in $\mathbb{R}^2$.

It is equal to $[f:4]$ where f is the linear functional $f(x,y)=x-y$,

i.e. the set of points in $\mathbb{R}^2$ at which $f(x,y)$ has the value 4.



Here, the standard matrix of the linear map $f: \mathbb{R}^2 \to \mathbb{R}^1$ is

$$A = [1 \ \ -1], \quad \text{a } 1\times 2 \text{ matrix}$$

Discussion

In general, take a nonzero linear functional $f : \mathbb{R}^n \to \mathbb{R}$.

Then the standard matrix of f is a $1 \times n$ non-zero matrix

$$A = [a_1 \ a_2 \ \cdots \ a_n]$$

(1) • So $[f : 0]$ is the same as $\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0\} = \text{Nul } A$

- Note $\text{rank } A = 1$ because there is exactly one pivot position,

  so $\dim \text{Nul } A = \underbrace{n}_{\#\text{ of cols}} - 1$ by the Rank (Nullity) Thm (Sec 4.5)

- So $[f : 0]$ is a subspace of $\mathbb{R}^n$ w/ dimension $n-1$.
- In particular, $[f : 0]$ is a flat in $\mathbb{R}^n$ w/ dimension $n-1$.
- Conclusion: $[f : 0]$ is a hyperplane in $\mathbb{R}^n$.

(2) • Now, let d be any number in $\mathbb{R}$.

- Then $[f : d]$ is the same as $\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = d\}$

  Recall fact:

  The set of solutions of the non-homogeneous system $A\vec{x} = \vec{b}$ can be obtained by translating the solution set of the homogeneous system $A\vec{x} = \vec{0}$ by any particular solution $\vec{p}$ of $A\vec{x} = \vec{b}$, i.e.

  $\{\text{solutions of } A\vec{x} = \vec{b}\} = \{\text{solutions of } A\vec{x} = 0\} + \vec{p}$

  where $\vec{p}$ is any solution of $A\vec{x} = \vec{b}$

- So $[f : d] = [f : 0] + \vec{p}$ for any $\vec{p}$ in $[f : d]$.
- Then $[f : d]$ is a flat in $\mathbb{R}^n$ w/ dimension $n-1$ parallel to $[f : 0]$
- Conclusion: $[f : d]$ is a hyperplane in $\mathbb{R}^n$ parallel to $[f : 0]$

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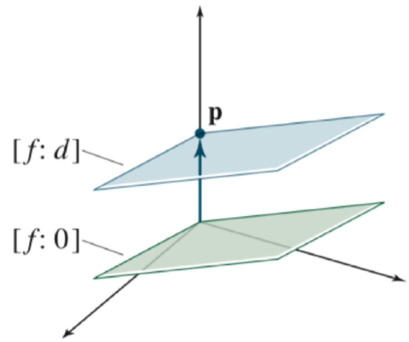


FIGURE 1 Parallel hyperplanes, with $f(\mathbf{p}) = d$.

(3) • Since $A = [a_1 \ a_2 \ \cdots \ a_n]$ is a $1 \times n$ matrix, the equation

$$A\vec{x} = d$$

$$[a_1 \ a_2 \ \cdots \ a_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = d$$

may be written w/ an inner product using $\vec{n} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ (the transpose of A)

$$\vec{n} \cdot \vec{x} = d$$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = d$$

• Thus $[f:d] = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = d\}$ is the same as $\{\vec{x} \in \mathbb{R}^n \mid \vec{n} \cdot \vec{x} = d\}$

and $[f:0] = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0\}$ is the same as $\{\vec{x} \in \mathbb{R}^n \mid \vec{n} \cdot \vec{x} = 0\}$.

• Observe $\underbrace{[f:0] = \{\vec{x} \in \mathbb{R}^n \mid \vec{n} \cdot \vec{x} = 0\} \text{ is the orthogonal complement of } \operatorname{Span}\{\vec{n}\}}$.

using $\mathbb{R}^3$ calculus / geometry terminology, say "$\vec{n}$ is a normal vector to $[f:0]$"

• We still say $\vec{n}$ is normal to each parallel hyperplane $[f:d]$

although $\vec{n} \cdot \vec{x}$ is not zero when $d \neq 0$

<u>Book Ex 3:</u> Let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; \vec{n} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 12 \right\}$ where $\vec{n} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

a) Find an implicit equation of H (i.e. a linear equation $ax+by=c$)

<u>Sol:</u> $\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 12$, so an implicit equation of H is $\boxed{3x + 4y = 12}$

b) Find a linear functional f and a scalar d such that $H = [f : d]$

<u>Sol:</u> $\boxed{f(x,y) = 3x + 4y}$ and $\boxed{d = 12}$

c) Let $\vec{v} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$. Consider the parallel hyperplane (line) $H_1 = H + \vec{v}$.

Find an implicit description of H_1.

<u>Sol:</u>

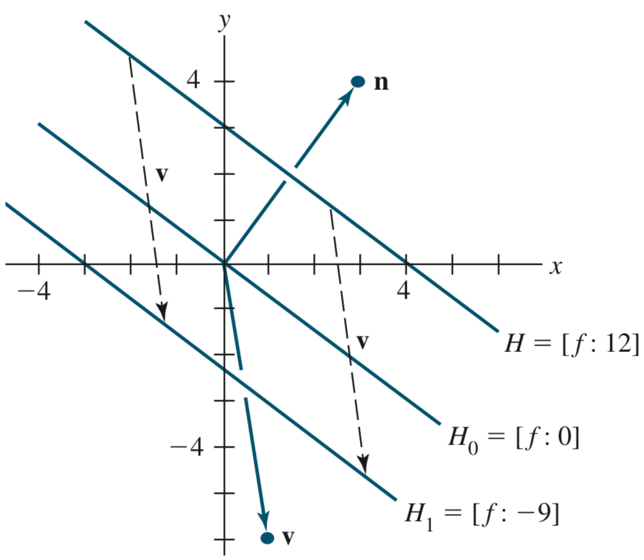
H_1 will be of the form $[f : d_1]$ for some d_1 in $\mathbb{R}$.

To find d_1, first find a point $\vec{p}$ in H_1
(by taking a point in H and adding $\vec{v}$ to it).

A convenient point in H to work with is $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

Let $\vec{p} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Compute $\vec{n} \cdot \vec{p} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix} = 3 - 12 = -9$. Then $d_1 = -9$, so $\boxed{H_1 = [f : -9]}$.



$3x + 4y = 12$

$3x + 4y = 0$

$3x + 4y = -9$

d) Give an explicit description of the line $3x+4y=12$ in parametric vector form, i.e. of form $\begin{bmatrix} x \\ y \end{bmatrix} = \vec{v}_1 + t\,\vec{v}_2$, t in $\mathbb{R}$.

Sol: Solve the equation $\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \end{bmatrix}$

$A \quad \vec{x} = \vec{b}$

The matrix $\begin{bmatrix} 3 & 4 \end{bmatrix}$ is already in (row) echelon form.

Let y be the free variable.

$3x+4y=12 \Rightarrow x = \frac{12}{3} - \frac{4y}{3}$

The solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 - \frac{4}{3}y \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - y \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$, y in $\mathbb{R}$

An explicit description of $3x+4y=12$ in parametric form is

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}, \text{ for } t \text{ in } \mathbb{R}}$$

EXTRA EXAMPLE

Converting an explicit description of a line into implicit form is more involved. The basic idea is to construct $[f:0]$ and then find d for $[f:d]$.

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EXAMPLE 5 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and let L_1 be the line through $\mathbf{v}_1$ and $\mathbf{v}_2$. Find a linear functional f and a constant d such that $L_1 = [f:d]$.

SOLUTION The line L_1 is parallel to the translated line L_0 through $\mathbf{v}_2 - \mathbf{v}_1$ and the origin. The defining equation for L_0 has the form

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \text{or} \quad \mathbf{n}\cdot\mathbf{x} = 0, \quad \text{where} \quad \mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{5}$$

Since $\mathbf{n}$ is orthogonal to the subspace L_0, which contains $\mathbf{v}_2 - \mathbf{v}_1$, compute

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 6 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

and solve

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = 0$$

By inspection, a solution is $\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} 2 & 5 \end{bmatrix}$. Let $f(x, y) = 2x + 5y$. From (5), $L_0 = [f:0]$, and $L_1 = [f:d]$ for some d. Since $\mathbf{v}_1$ is on line L_1, $d = f(\mathbf{v}_1) = 2(1) + 5(2) = 12$. Thus, the equation for L_1 is $2x + 5y = 12$. As a check, note that $f(\mathbf{v}_2) = f(6, 0) = 2(6) + 5(0) = 12$, so $\mathbf{v}_2$ is on L_1, too. ■

<u>Book Ex 6</u> Let H_1 be the plane that contains the points $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

Find an implicit description $[f : d]$ of H_1.

(Note: Since we are in the special case of $\mathbb{R}^3$, we can use the cross-product formula (from Calc 3) to compute a normal vector for the plane.
But we'll use a procedure below that generalizes to higher dimension $n \geq 4$)

<u>Sol:</u>

(Step 1) Recall (from Sec 8.1 "Affine combinations" Thm 1)

H_1 is parallel to $H_0 = \text{Span}\left\{ \underbrace{\vec{v_2} - \vec{v_1}}_{\text{denote } \vec{p}}, \underbrace{\vec{v_3} - \vec{v_1}}_{\text{denote } \vec{q}} \right\} = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}}_{\vec{p}}, \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{q}} \right\}$

Note H_0 is a subspace, so H_0 is the plane containing $\vec{p}$, $\vec{q}$, and the origin.

(Step 2) Find a vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ which is normal to H_0:

Recall (from Sec 6.1):
A vector $\vec{z}$ in $\mathbb{R}^n$ is said to be <u>orthogonal</u> to a subspace W of $\mathbb{R}^n$ if $\vec{z}$ is orthogonal to <u>every</u> vector in W, but it's sufficient to check that $\vec{z}$ is orthogonal to <u>every</u> vector in a spanning set of W.

Let's find a nonzero $\vec{n}$ which is orthogonal to both $\vec{p}$ and $\vec{q}$.

Set $\vec{p} \cdot \vec{n} = 0$: $\quad 1a - 2b + 3c = 0$
Set $\vec{q} \cdot \vec{n} = 0$: $\quad 2a \quad\quad + c = 0$

$\left. \right\}$ Solve for a, b, c

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{Row reduce}} \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 4 & -5 & 0 \end{array}\right] \quad \left.\begin{array}{l} 2a + c = 0 \\ 4b - 5c = 0 \end{array}\right\} \quad \begin{array}{l} 2a = -c \\ 4b = 5c \end{array} \quad c \text{ is free}$$

Choose any nonzero c, let's say $c = 4$. So $a = -\frac{4}{2} = -2$, $b = \frac{5(4)}{4} = 5$

Let $\vec{n} = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$. Then $H_0 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \right\}$, i.e. $H_0 = [f : 0]$
where $f(x,y,z) = -2x + 5y + 4z$.

(Step 3) Find d where $H_1 = [f : d]$.

Simply choose a point in H_1 and input it into f:

We know $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are in H_1. Set $d = f(\vec{v_1}) = -2(1) + 5(1) + 4(1) = 7$.
(Check: $f(\vec{v_2}) = f(\vec{v_3}) = 7$ ✓)

Then $\boxed{H_1 \text{ is } [f : 7] \text{ where } f(x,y,z) = -2x + 5y + 4z.}$ or $\boxed{-2x + 5y + 4z = 7}$

In Summary ...

Thm (Thm 11) Suppose H is a subset of $\mathbb{R}^n$. Then TFAE:

(a) H is a hyperplane

(b) $H = [f : d]$ for some nonzero linear functional f and some d in $\mathbb{R}$

(c) $H = \{\vec{x} \text{ in } \mathbb{R}^n \mid \vec{n} \cdot \vec{x} = d\}$ for some nonzero vector $\vec{n}$ and some d in $\mathbb{R}$

— the end —