Sec 8.3 Convex combinations

Recall from Sec 8.1

Def • An <u>affine combination</u> of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ in $\mathbb{R}^n$ is a linear combination $c_1\vec{v}_1 + \dots + c_p\vec{v}_p$ such that $\underbrace{c_1 + c_2 + \dots + c_p = 1}_{\text{the weights sum up to } 1}$

• The <u>affine hull</u> (or <u>affine span</u>) of a set S is the set of all affine combinations of points in S.
(Notation: aff S)

Def (New) • A <u>convex combination</u> of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ in $\mathbb{R}^n$ is a linear combination $c_1\vec{v}_1 + \dots + c_p\vec{v}_p$ such that $\underbrace{c_1 + c_2 + \dots + c_p = 1}_{\text{the weights sum up to } 1}$

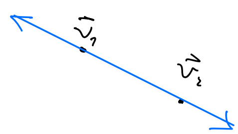
AND all $c_1, c_2, \dots, c_p$ are nonnegative

• The <u>convex</u> <u>hull</u> of a set S is the set of all convex combinations of points in S. (Notation: Conv S)

Note: • aff $\{\vec{v}\}$ is just $\{\vec{v}\}$,
conv $\{\vec{v}\}$ is also $\{\vec{v}\}$

• aff $\{\vec{v}_1, \vec{v}_2\}$ is $\{\overbrace{(1-t)\vec{v}_1 + t\vec{v}_2}^{\text{affine combinations of } \vec{v}_1 \text{ and } \vec{v}_2} \mid t \text{ in } \mathbb{R}\}$,
(distinct points)

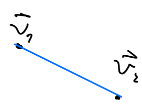
the line containing both $\vec{v}_1, \vec{v}_2$.



• conv $\{\vec{v}_1, \vec{v}_2\}$ is $\{\overbrace{(1-t)\vec{v}_1 + t\vec{v}_2}^{\text{convex combinations of } \vec{v}_1 \text{ and } \vec{v}_2} \mid t \text{ between } 0 \text{ and } 1\}$,

Because the weights in a convex combination are nonnegative

the line segment between $\vec{v}_1$ and $\vec{v}_2$, denoted $\overline{\vec{v}_1\vec{v}_2}$.



Recall (Sec 8.2 "Unique representation thm" & "barycentric coordinates")

If S is affinely independent and $\vec{p}$ is in aff S

then $\vec{p}$ can be written uniquely as an affine combination of S.

The weights are called barycentric coordinates of $\vec{p}$.

Note: $\vec{p}$ is in conv S iff the barycentric coordinates of $\vec{p}$ are nonnegative.

EXAMPLE 1 Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -6 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{p}_1 = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} -10 \\ 5 \\ 11 \\ -4 \end{bmatrix},$$

Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ & note S is an $\underbrace{\text{orthogonal}}_{\text{the points are pairwise orthogonal}}$ set.

Determine whether $\vec{p}_1, \vec{p}_2$ are in Span S, aff S, & conv S.

Sol: Let $W = \text{Span } S$

Recall that $\text{proj}_W \vec{p}$ is the closest point in W to $\vec{p}$,

so $\vec{p}$ is in W iff $\text{proj}_W \vec{p} = \vec{p}$.

For $\vec{p}_1$:

$$\text{proj}_W \vec{p}_1 = \text{proj}_{\vec{v}_1} \vec{p}_1 + \text{proj}_{\vec{v}_2} \vec{p}_1 + \text{proj}_{\vec{v}_3} \vec{p}_1$$

(We can do this because $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis of W, see Sec 6.2 & 6.3)

$$= \frac{\mathbf{p}_1 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{p}_1 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{p}_1 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3$$

$$= \frac{18}{54}\mathbf{v}_1 + \frac{18}{54}\mathbf{v}_2 + \frac{18}{54}\mathbf{v}_3$$

$$= \frac{1}{3}\begin{bmatrix} 3 \\ 0 \\ 6 \\ -3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -6 \\ 3 \\ 3 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 3 \\ 6 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix} = \mathbf{p}_1$$

This shows $\vec{p}_1$ is in $W = \text{Span } S$

Since the coefficients sum up 1, $\vec{p}_1$ is in aff S.

______ " ______ and are nonnegative, $\vec{p}_1$ is in conv S.

For $\vec{p}_2$: $\quad \text{proj}_W \vec{p}_2 = \dfrac{\vec{p}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \dfrac{\vec{p}_2 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 + \dfrac{\vec{p}_2 \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} \vec{v}_3$

$$= \frac{48}{54} \vec{v}_1 + \frac{108}{54} \vec{v}_2 + \frac{-12}{54} \vec{v}_3$$

$$= \frac{8}{9} \begin{bmatrix} 3 \\ 0 \\ 6 \\ -3 \end{bmatrix} + 2 \begin{bmatrix} -6 \\ 3 \\ 3 \\ 0 \end{bmatrix} - \frac{2}{9} \begin{bmatrix} 3 \\ 6 \\ 0 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} * \\ 0 + 2(3) - \frac{12}{9} \\ * \\ * \end{bmatrix} \neq \vec{p}_2$$

So $\vec{p}_2$ is not in $W = \text{Span}\, S$.

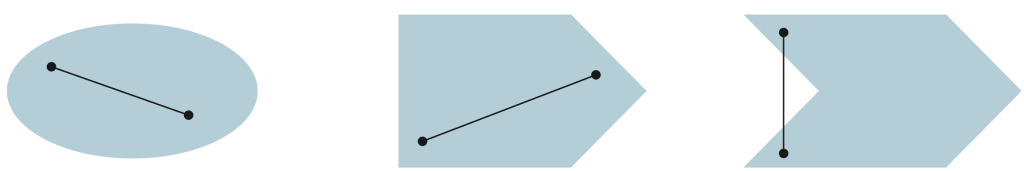
Thus $\vec{p}_2$ cannot be in aff S or conv S.

Recall (Sec 8.1): A set S is called <u>affine</u> if:

$\vec{p}, \vec{q}$ in S implies the entire line through $\vec{p}$ & $\vec{q}$ is in S.

DEFINITION A set S is **convex** if for each $\mathbf{p}, \mathbf{q} \in S$, the line segment $\overline{\mathbf{pq}}$ is contained in S.

Intuitively, a set S is convex if every two points in the set can "see" each other without the line of sight leaving the set. Figure 1 illustrates this idea.



Convex Convex Not convex

THEOREM 7

A set S is convex if and only if every convex combination of points of S lies in S. That is, S is convex if and only if $S = \text{conv}\, S$.

THEOREM 8

The intersection of two subspaces is a subspace

— ″ — of two convex sets is a convex set

— ″ — of two affine sets is an affine set

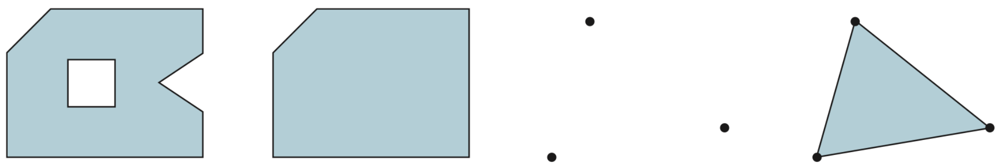
THEOREM 9

For any set S, the convex hull of S is the intersection of all the convex sets that contain S.

i.e. conv S is the "smallest" convex set containing S.

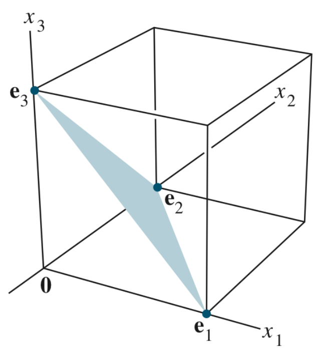
EXAMPLE 2

a. The convex hulls of sets S and T in $\mathbb{R}^2$ are shown below.



S conv S T conv T

b. Let S be the set consisting of the standard basis for $\mathbb{R}^3$, $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then conv S is a triangular surface in $\mathbb{R}^3$, with vertices $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. See Figure 2. ∎



If $\vec{p}$ is in the convex hull of S, then (by def) $\vec{p}$ is a convex combination of points of S.

How many points are needed for this combination?

THEOREM 10

(Caratheodory) If S is a nonempty subset of $\mathbb{R}^n$, then every point in conv S can be expressed as a convex combination of $n+1$ or fewer points of S.