

Sec 7.4 The singular value decomposition (SVD)

Goal: Not all matrices can be factored as $A = \underbrace{P D P^{-1}}_{\text{diagonal}}$,

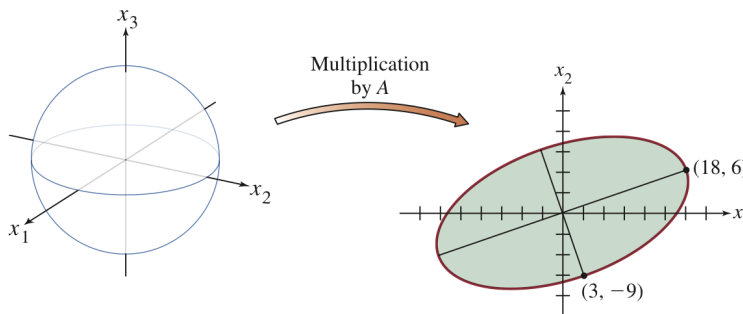
but every $m \times n$ matrix can be factored as $A = Q \Sigma P^{-1}$.

Recall (Ch 5) if λ_1 is an eigenvalue of $n \times n$ A w/ the greatest length, then a corresponding unit eigenvector $\vec{v}_1$ gives the direction in which the stretching effect of A is greatest:

$$\|A \vec{v}_1\| = \|\lambda_1 \vec{v}_1\|$$

We can generalize this concept to $m \times n$ matrices, & this will lead to the SVD.

Ex 1: $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.



The linear map $\vec{x} \mapsto A \vec{x}$ maps the unit sphere in $\mathbb{R}^3$ onto an ellipse in $\mathbb{R}^2$.

Problem: find a unit vector at which the length $\|A \vec{x}\|$ is maximized & compute $\|A \vec{x}\|$.

Sol:

Note: • $\|A \vec{x}\|^2$ is maximized at the same $\vec{x}$ that maximizes $\|A \vec{x}\|$.

• $\|A \vec{x}\|^2 = A \vec{x} \cdot A \vec{x} = (A \vec{x})^T (A \vec{x}) = \vec{x}^T A^T (A \vec{x}) = \vec{x}^T \underbrace{(A^T A)}_{\substack{\text{def of} \\ \text{inner} \\ \text{product}}} \vec{x}$

• $A^T A$ is a symmetric matrix, since $(A^T A)^T = A^T (A^T)^T = A^T A$

optional
intro

This page is part of optional review

Facts Let B be a symmetric matrix.

(Thm 6 in Sec 7.3) Let M be the largest possible value $\vec{x}^T B \vec{x}$ for unit vectors $\vec{x}$. (Note: M is a real number)

Then: ① M = the largest eigenvalue λ of B

② If $\vec{v}$ is a unit eigenvector of λ , then $\vec{v}^T B \vec{v} = M$.

- So the maximum value M of $\|A\vec{x}\|^2$ is equal to the greatest eigenvalue λ of $B = A^T A$. And the maximum value is attained at a unit eigenvector of $B = A^T A$ corresponding to λ .

For this problem, compute

$$A^T A = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

The eigenvalues of $A^T A$ are $\lambda_1 = 360$, $\lambda_2 = 90$, and $\lambda_3 = 0$. Corresponding unit eigenvectors are, respectively,

$$\mathbf{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

The maximum value of $\|A\mathbf{x}\|^2$ is 360, attained when $\mathbf{x}$ is the unit vector $\mathbf{v}_1$. The vector $A\mathbf{v}_1$ is a point on the ellipse in Figure 1 farthest from the origin, namely

$$A\mathbf{v}_1 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$$

For $\|\mathbf{x}\| = 1$, the maximum value of $\|A\mathbf{x}\|$ is $\|A\mathbf{v}_1\| = \sqrt{360} = 6\sqrt{10}$. ■

Beginning of lecture about SVD

The singular values of an $m \times n$ matrix

Let A be an $m \times n$ matrix.

Then $A^T A$ is symmetric, since $\underbrace{(A^T A)^T}_{B^T} = A^T \underbrace{(A^T)^T}_B = A^T A$

and so, by Sec 7.1, $A^T A$ can be orthogonally diagonalized.

Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$

consisting of eigenvectors of $A^T A$, and

let $\lambda_1, \dots, \lambda_n$ be the associated eigenvalues of $A^T A$.

Then, for each eigenvector $\vec{v}_i$,

$$\begin{aligned} \|A\vec{v}_i\|^2 &= \underbrace{(A\vec{v}_i)^T (A\vec{v}_i)}_{\text{def of inner product}} = \vec{v}_i^T \underbrace{A^T A}_{\lambda_i \vec{v}_i} \vec{v}_i && \text{Since } \vec{v}_i \text{ is a } \lambda_i\text{-eigenvector of } A^T A \\ &= \vec{v}_i^T \lambda_i \vec{v}_i \\ &= \lambda_i \vec{v}_i^T \vec{v}_i \\ &= \lambda_i \|\vec{v}_i\|^2 \\ &= \lambda_i \underbrace{(1)}_{\text{since } \vec{v}_i \text{ is a unit vector}} \end{aligned}$$

$$\text{Thus } \|A\vec{v}_i\| \stackrel{(*)}{=} \sqrt{\lambda_i}$$

Observe: The eigenvalues of $A^T A$ are all nonnegative.

Def

The **singular values** of A are the square roots of the eigenvalues of $A^T A$, denoted by $\sigma_1, \dots, \sigma_n$, and they are arranged in decreasing order. That is, $\sigma_i = \sqrt{\lambda_i}$ for $1 \leq i \leq n$. By equation (*), the singular values of A are the lengths of the vectors $A\mathbf{v}_1, \dots, A\mathbf{v}_n$.

Ex Find the singular values of $A = \begin{bmatrix} 7 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}$

$$\text{Sol: } A^T A = \begin{bmatrix} 7 & 0 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 65 & 30 \\ 30 & 20 \end{bmatrix}$$

• Find the eigenvalues of $A^T A$:

$$\begin{aligned} \det(A^T A - \lambda I) &= \det \begin{pmatrix} 65-\lambda & 30 \\ 30 & 20-\lambda \end{pmatrix} \\ &= (65-\lambda)(20-\lambda) - 30^2 \\ &= \lambda^2 - 85\lambda + 1300 - 900 \\ &= \lambda^2 - 85\lambda + 400 \\ &= (\lambda - 80)(\lambda - 5) \end{aligned}$$

Eigenvalues of $A^T A$ are $\lambda_1 = 80$, $\lambda_2 = 5$

• The singular values of A are $\sigma_1 > \sigma_2$
 $4\sqrt{5} = \sqrt{80} > \sqrt{5}$

Note: If v_1 and v_2 are unit eigenvectors corresponding to λ_1, λ_2 ,
 then

$$\|A \hat{v}_i\| \stackrel{(*)}{=} \sqrt{\lambda_i} = \sigma_i$$

(Thm 9) Suppose $\{\hat{v}_1, \dots, \hat{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, arranged so that the corresponding eigenvalues of $A^T A$ satisfy

$$\lambda_1 \geq \dots \geq \lambda_n.$$

Let $r = \#$ of nonzero singular values of A .

Then: • $\text{rank } A = r$

• $\{A \hat{v}_1, \dots, A \hat{v}_r\}$ is an orthogonal basis for $\text{Col } A$.

(Partial Proof) $(A \hat{v}_1) \cdot (A \hat{v}_2) = (A \hat{v}_1)^T A \hat{v}_2 = \hat{v}_1^T A^T A \hat{v}_2 = \hat{v}_1^T \lambda_2 \hat{v}_2 = \lambda_2 \hat{v}_1^T \hat{v}_2 = 0$
 since $\hat{v}_1, \hat{v}_2$ are orthogonal

Numerical Notes

In some cases, the rank of A may be very sensitive to small changes in the entries of A . The obvious method of counting the number of pivot columns in A does not work well if A is row reduced by a computer. Roundoff error often creates an echelon form with full rank.

In practice, the most reliable way to estimate the rank of a large matrix A is to count the number of nonzero singular values. In this case, extremely small nonzero singular values are assumed to be zero for all practical purposes, and the *effective rank* of the matrix is the number obtained by counting the remaining nonzero singular values.¹

The Singular Value Decomposition

The decomposition of A involves an $m \times n$ “diagonal” matrix Σ of the form

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} \text{--- } n \text{ ---} \\ \text{--- } m \text{ ---} \end{array} \quad \begin{array}{l} \leftarrow m - r \text{ rows} \\ \uparrow n - r \text{ columns} \end{array}$$

Theorem 10 (The Singular value decomposition)

Let A be an $m \times n$ matrix w/ rank r .

Then there exist :

(1) an $m \times n$ matrix
same size as A

$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_r > 0$ are the first r singular values of A

(2) an $m \times m$ orthogonal matrix U

(3) an $n \times n$ orthogonal matrix V

such that

$$A = U \Sigma V^T$$

The columns of U are called left singular vectors of A

- The columns of V are called right singular vectors of A

- Any such factorization is called a singular value decomposition of A .
- The diagonal entries of Σ are necessarily the singular values of A (so Σ is unique)
- The matrices U and V are not uniquely determined by A .

How to find a singular value decomposition of A (that is also a proof for Thm 10 (SVD)):

(1) Find the eigenvalues of $A^T A$ $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ (in descending order)

Take square roots to get the nonzero singular values

$$\begin{array}{ccc} \sqrt{\lambda_1} & \sqrt{\lambda_2} & \dots & \sqrt{\lambda_r} \\ \parallel & & & \\ \sigma_1 & \geq \sigma_2 & \geq \dots & \geq \sigma_r \end{array}$$

Let $D = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$, Let $\Sigma = \left[\begin{array}{ccc|c} \sigma_1 & \sigma_2 & \dots & 0 \\ & & & \\ & & & \\ \hline & & & 0 \end{array} \right]$ Size $m \times n$
Same size as A

(2) Produce an orthogonal set of eigenvectors of $A^T A$

associated to $\lambda_1, \lambda_2, \dots, \lambda_n$ (possibly applying the Gram-Schmidt process if there are repeated eigenvalues).

Normalize them to find unit eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^n$

Let $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix}$
 $n \times n$

(3) Compute $A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_r \in \mathbb{R}^m$ only for eigenvectors corresponding to nonzero eigenvalues $\lambda_1, \dots, \lambda_r$.

Normalize them $\vec{u}_1 := \frac{1}{\sigma_1} A\vec{v}_1$, $\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2$, $\dots$, $\vec{u}_r = \frac{1}{\sigma_r} A\vec{v}_r$.

Since $\|A\vec{v}_i\|^2 = \lambda_i$ as we showed earlier in eq (*), $\|A\vec{v}_i\| = \sqrt{\lambda_i} = \sigma_i$

By Thm 9, $\{\vec{u}_1, \dots, \vec{u}_r\}$ is an orthogonal set (so it's orthonormal).

Complete to an orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_r, \dots, \vec{u}_m\}$ for $\mathbb{R}^m$

Let $U = \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_m \end{bmatrix}$

Note

$$U\Sigma = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_m] \left[\begin{array}{cccc|c} \sigma_1 & & & & 0 \\ & \sigma_2 & & & \\ & & \ddots & & \\ 0 & & & \sigma_r & \\ \hline & 0 & & & 0 \end{array} \right]$$

$$= [\sigma_1 \mathbf{u}_1 \quad \cdots \quad \sigma_r \mathbf{u}_r \quad \mathbf{0} \quad \cdots \quad \mathbf{0}]$$

$$= AV$$

↖ because $\sigma_1 \hat{\mathbf{u}}_1 = A \hat{\mathbf{v}}_1$, $\sigma_2 \hat{\mathbf{u}}_2 = A \hat{\mathbf{v}}_2$, ..., $\sigma_r \hat{\mathbf{u}}_r = A \hat{\mathbf{v}}_r$

$$\text{So } U\Sigma V^T = AVV^T$$

$$= A \quad \text{since } V \text{ is orthogonal}$$

— end of algorithm (also proof) for SVD —

Ex of SVD

$$\text{Let } A = \begin{bmatrix} 7 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} \text{ as before}$$

$$\left(\begin{array}{l} \text{Step 1) } \\ \text{From} \\ \text{prev} \\ \text{ex} \end{array} \right\} \begin{array}{l} A^T A = \begin{bmatrix} 7 & 0 & 4 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 65 & 30 \\ 30 & 20 \end{bmatrix} \text{ has eigenvalues } \lambda_1 = 80, \lambda_2 = 5 \\ \text{So the singular values of } A \text{ are } \sigma_1 = \sqrt{80} = 4\sqrt{5}, \sigma_2 = \sqrt{5} \end{array}$$

$$\text{Let } D = \begin{bmatrix} 4\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}, \text{ let } \Sigma = \begin{bmatrix} 4\sqrt{5} & 0 \\ 0 & \sqrt{5} \\ 0 & 0 \end{bmatrix} \quad \text{size } 3 \times 2$$

(Step 2)

Find the eigenvectors of $A^T A$ associated to $\lambda_1=80$, $\lambda_2=5$

For $\lambda_1=80$: Find basis for $\text{Nul}(A^T A - 80 \text{Id}) = \text{Nul}\left(\begin{bmatrix} 65-80 & 30 \\ 30 & 20-80 \end{bmatrix}\right)$

$$\left[\begin{array}{cc|c} -15 & 30 & 0 \\ 30 & -60 & 0 \end{array}\right] \xrightarrow{\text{Row reduce}} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_1 - 2x_2 = 0$$

x_2 can be any nonzero number, so let $x_2 = 1$. Then $x_1 = 2$

An 80-eigenvector for $A^T A$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\text{Normalize: } \vec{v}_1 = \frac{1}{\sqrt{2^2+1}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

For $\lambda_2=5$: Find basis for $\text{Nul}(A^T A - 5 \text{Id}) = \text{Nul}\left(\begin{bmatrix} 65-5 & 30 \\ 30 & 20-5 \end{bmatrix}\right)$

$$\left[\begin{array}{cc|c} 60 & 30 & 0 \\ 30 & 15 & 0 \end{array}\right] \xrightarrow{\text{Row reduce}} \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} 2x_1 + x_2 = 0 \\ 2x_1 = -x_2 \\ x_1 = -\frac{x_2}{2} \end{array}$$

Let $x_2 = 2$. Then $x_1 = -1$.

So a 5-eigenvector for $A^T A$ is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\text{Normalize: } \vec{v}_2 = \frac{1}{\sqrt{1^2+2^2}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\text{Let } V = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}, \text{ then } V^T = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

(Step 3) Compute $A\vec{v}_1, A\vec{v}_2$

$$A\vec{v}_1 = \begin{bmatrix} 7 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 16/\sqrt{5} \\ 0 \\ 12/\sqrt{5} \end{bmatrix}$$

$$A\vec{v}_2 = \begin{bmatrix} 7 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -3/\sqrt{5} \\ 0 \\ 4/\sqrt{5} \end{bmatrix}$$

Normalize: $\vec{u}_1 := \frac{1}{\sqrt{4}} A\vec{v}_1 = \frac{1}{4\sqrt{5}} \begin{bmatrix} 16/\sqrt{5} \\ 0 \\ 12/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4/5 \\ 0 \\ 3/5 \end{bmatrix}$

Since $\|A\vec{v}_1\| = \sqrt{\lambda_1}$

Normalize: $\vec{u}_2 = \frac{1}{\sqrt{2}} A\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -3/\sqrt{5} \\ 0 \\ 4/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -3/5 \\ 0 \\ 4/5 \end{bmatrix}$

$S = \{\vec{u}_1, \vec{u}_2\}$ is a linearly independent set in $\mathbb{R}^3$.

Extend S to a basis of $\mathbb{R}^3$ by finding $\vec{u}_3$

such that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is orthogonal.

We want $\vec{u}_1 \cdot \vec{x} = 0$ and $\vec{u}_2 \cdot \vec{x} = 0$

Set $\frac{1}{5} \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ and $\frac{1}{5} \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

So $4x_1 + 3x_3 = 0$ and $-3x_1 + 4x_3 = 0$

$$\left[\begin{array}{ccc|c} 4 & 0 & 3 & 0 \\ -3 & 0 & 4 & 0 \end{array} \right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 12 & 0 & 9 & 0 \\ -12 & 0 & 16 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 12 & 0 & 9 & 0 \\ 0 & 0 & 25 & 0 \end{array} \right]$$

x_2 can be any number

$$\left. \begin{array}{l} 12x_1 + 9x_3 = 0 \\ 25x_3 = 0 \end{array} \right\} \begin{array}{l} x_1 = 0 \\ x_3 = 0 \end{array}$$

We can choose $\vec{u}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, already a unit vector.

Alternatively, note that both $\vec{u}_1, \vec{u}_2$ have 0 in 2nd entry,

so I could have guessed $\vec{u}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ would work

Let $U = \begin{bmatrix} 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \\ 3/5 & 4/5 & 0 \end{bmatrix}$

$$U \Sigma V^T = \begin{bmatrix} 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \\ 3/5 & 4/5 & 0 \end{bmatrix} \begin{bmatrix} 4\sqrt{5} & 0 \\ 0 & \sqrt{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

is a singular value decomposition of $A = \begin{bmatrix} 7 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}$
 — end of example —

Bases for fundamental subspaces (see Example 6 in book)

Perform an SVD for an $m \times n$ matrix A . Let $r = \text{rank } A$.

1. $\{A\vec{v}_1, \dots, A\vec{v}_r\}$ is an orthogonal basis for $\text{Col } A$ (Thm 9)

So $\{\text{the first } r \text{ columns of } U\} = \{\vec{u}_1, \dots, \vec{u}_r\}$

is an orthonormal basis for $\text{Col } A$

also called image or range of A

2. Also, it's a fact that $(\text{Col } A)^\perp = \text{Nul}(A^T)$

So $\{\text{the rest of the columns of } U\} = \{\vec{u}_{r+1}, \dots, \vec{u}_m\}$

is an orthonormal basis for $\text{Nul}(A^T)$

also called cokernel of A

3. Since $\|A\vec{v}_i\| = \sigma_i$ for $1 \leq i \leq n$ and $\sigma_i = 0$ iff $i > r$,

$\{\vec{v}_{r+1}, \dots, \vec{v}_n\} = \{\text{the last } n-r \text{ columns of } V\}$

is an orthonormal basis for $\text{Nul } A$

also called the kernel of A

4. We also have $(\text{Nul } A)^\perp = \text{Row } A$

So $\{\vec{v}_1, \dots, \vec{v}_r\} = \{\text{the first } r \text{ columns of } V\}$

is an orthonormal basis for $\text{Row } A$.

— we ended here on Thurs —

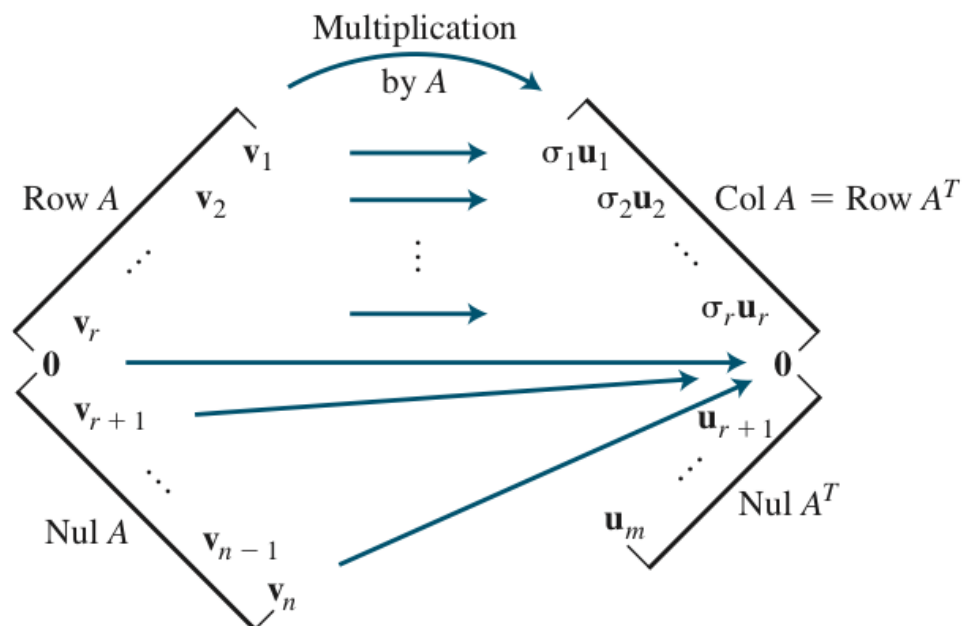


FIGURE 4 The four fundamental subspaces and the action of A .

The Invertible matrix theorem (the end of the list)

Let A be an $n \times n$ matrix. TFAE:

(a) A is invertible.

(b) $\vdots$

(c) $\vdots$

.

(s) $(\text{Col } A)^\perp = \{\vec{0}\}$

(t) $(\text{Nul } A)^\perp = \mathbb{R}^n$

(u) $\text{Row } A = \mathbb{R}^n$

(v) A has n nonzero singular values

Additional
Ex of SVD
from book

EXAMPLE 4 Find a singular value decomposition of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.

SOLUTION First, compute $A^T A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$. The eigenvalues of $A^T A$ are 18 and 0, with corresponding unit eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

These unit vectors form the columns of V :

$$V = [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

The singular values are $\sigma_1 = \sqrt{18} = 3\sqrt{2}$ and $\sigma_2 = 0$. Since there is only one nonzero singular value, the “matrix” D may be written as a single number. That is, $D = 3\sqrt{2}$. The matrix Σ is the same size as A , with D in its upper left corner:

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

To construct U , first construct $A\mathbf{v}_1$ and $A\mathbf{v}_2$:

$$A\mathbf{v}_1 = \begin{bmatrix} 2/\sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix}, \quad A\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As a check on the calculations, verify that $\|A\mathbf{v}_1\| = \sigma_1 = 3\sqrt{2}$. Of course, $A\mathbf{v}_2 = \mathbf{0}$ because $\|A\mathbf{v}_2\| = \sigma_2 = 0$. The only column found for U so far is

$$\mathbf{u}_1 = \frac{1}{3\sqrt{2}} A\mathbf{v}_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

The other columns of U are found by extending the set $\{\mathbf{u}_1\}$ to an orthonormal basis for $\mathbb{R}^3$. In this case, we need two orthogonal unit vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ that are orthogonal to $\mathbf{u}_1$. (See Figure 3.) Each vector must satisfy $\mathbf{u}_i^T \mathbf{x} = 0$, which is equivalent to the equation $x_1 - 2x_2 + 2x_3 = 0$. A basis for the solution set of this equation is

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(Check that $\mathbf{w}_1$ and $\mathbf{w}_2$ are each orthogonal to $\mathbf{u}_1$.) Apply the Gram–Schmidt process (with normalizations) to $\{\mathbf{w}_1, \mathbf{w}_2\}$, and obtain

$$\mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

Finally, set $U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$, take Σ and V^T from above, and write

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

■

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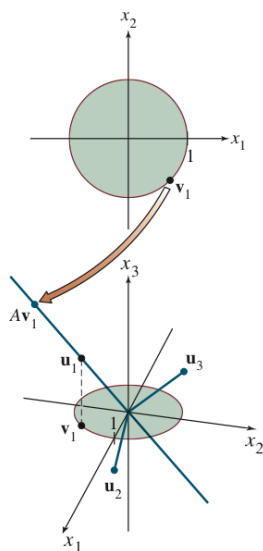


FIGURE 3