

Recall (From Sec 6.2)

Def An orthogonal matrix is a square invertible matrix U such that $U^{-1} = U^T$.

Theorem Let U be an $n \times n$ matrix.

TFAE:

- (1) U is an orthogonal matrix
- (2) The columns of U form an orthonormal set

7.1 Diagonalization of Symmetric Matrices

A **symmetric** matrix is a matrix A such that $A^T = A$. Such a matrix is necessarily square. Its main diagonal entries are arbitrary, but its other entries occur in pairs—on opposite sides of the main diagonal.

EXAMPLE 1 Of the following matrices, only the first three are symmetric:

$$\text{Symmetric: } \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & -7 \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

$$\text{Nonsymmetric: } \begin{bmatrix} 1 & -3 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -4 & 0 \\ -6 & 1 & -4 \\ 0 & -6 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \quad \blacksquare$$

To begin the study of symmetric matrices, it is helpful to review the diagonalization process of Section 5.3.

(Additional
Ex)

EXAMPLE 2 If possible, diagonalize the matrix $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$.

SOLUTION The characteristic equation of A is $0 = \det(A - \lambda I)$

$$0 = -\lambda^3 + 17\lambda^2 - 90\lambda + 144 = -(\lambda - 8)(\lambda - 6)(\lambda - 3)$$

Find a basis for each eigenspace using Sec 5.1 method
(same as finding basis for a null space)

$$\lambda = 8: \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = 6: \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}; \quad \lambda = 3: \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

These three vectors form a basis for $\mathbb{R}^3$. In fact, it is easy to check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an *orthogonal* basis for $\mathbb{R}^3$. Experience from Chapter 6 suggests that an *orthonormal* basis might be useful for calculations, so here are the normalized (unit) eigenvectors.

$$\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Let

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Then $A = PDP^{-1}$, as usual. But this time, since P is square and has orthonormal columns, P is an *orthogonal* matrix, and P^{-1} is simply P^T . (See Section 6.2.) ^{Thm 6} $\blacksquare$

Theorem 1:

If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

PROOF Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors that correspond to distinct eigenvalues, say, λ_1 and λ_2 . To show that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, compute

$$\begin{aligned}\lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2 &= (\lambda_1 \mathbf{v}_1)^T \mathbf{v}_2 = (A \mathbf{v}_1)^T \mathbf{v}_2 && \text{Since } \mathbf{v}_1 \text{ is an eigenvector} \\ &= (\mathbf{v}_1^T A^T) \mathbf{v}_2 = \mathbf{v}_1^T (A \mathbf{v}_2) && \text{Since } A^T = A \\ &= \mathbf{v}_1^T (\lambda_2 \mathbf{v}_2) && \text{Since } \mathbf{v}_2 \text{ is an eigenvector} \\ &= \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 = \lambda_2 \mathbf{v}_1 \cdot \mathbf{v}_2\end{aligned}$$

Hence $(\lambda_1 - \lambda_2) \mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. But $\lambda_1 - \lambda_2 \neq 0$, so $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. ■

Def: An $n \times n$ matrix is called orthogonally diagonalizable if we can write

$$A = P D P^T = P D P^{-1}$$

where (1) D is a diagonal matrix and

(2) the columns of P are orthonormal

Theorem 2

An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.

This theorem is rather amazing, because the work in Chapter 5 would suggest that it is usually impossible to tell when a matrix is diagonalizable. But this is not the case for symmetric matrices.

Proof (forward direction):

$$\text{Suppose } A = P D P^T = P D P^{-1}$$

$$\text{Then } A^T = (P D P^T)^T = P^{TT} D^T P^T = P D P^T = A$$

Thus A is symmetric! □

(Proof of the reverse direction is much harder.)

The next example treats a matrix whose eigenvalues are not all distinct.

EXAMPLE 3 Orthogonally diagonalize the matrix $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, whose characteristic equation is

$$0 = -\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 7)^2(\lambda + 2)$$

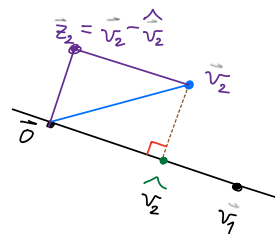
SOLUTION The usual calculations produce bases for the eigenspaces:

$$\lambda = 7: \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = -2: \mathbf{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

Recall:

not orthogonal, $\vec{v}_1 \cdot \vec{v}_2 \neq 0$

- If we start from a linearly independent set S , we can always construct an orthogonal set using the Gram-Schmidt process (Sec 6.4).
- If S has only two vectors $\vec{v}_1$ and $\vec{v}_2$, then:
the projection of $\vec{v}_2$ onto $\vec{v}_1$ is $\hat{\vec{v}}_2 = \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$ and
the component of $\vec{v}_2$ orthogonal to $\vec{v}_1$ is $\vec{z}_2 = \vec{v}_2 - \hat{\vec{v}}_2$



$$\bullet \text{ Let } \mathbf{z}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1/2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

• Note $\vec{z}_2$ is in $\text{Span}\{\vec{v}_2, \vec{v}_1\} = 7\text{-eigenspace}$.

• Then $\{\mathbf{v}_1, \mathbf{z}_2\}$ is an orthogonal set in the eigenspace for $\lambda = 7$.

• So $\{\vec{v}_1, \vec{z}_2\}$ is linearly independent by Thm 4 (Sec 6.2)

- So $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for the 7-eigenspace by the Basis Thm (Sec 4.5), since the 7-eigenspace has $\dim 2$.

Normalize $\mathbf{v}_1$ and $\mathbf{z}_2$ to obtain the following orthonormal basis for the eigenspace for $\lambda = 7$:

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

- An orthonormal basis for the eigenspace for $\lambda = -2$ is

$$\mathbf{u}_3 = \frac{1}{\|2\mathbf{v}_3\|} 2\mathbf{v}_3 = \frac{1}{3} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

- By Theorem 1, $\mathbf{u}_3$ is orthogonal to the other eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Hence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set. Let

$$P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}, \quad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Then P orthogonally diagonalizes A , and $A = PDP^{-1}$. ■

The Spectral Theorem

The set of eigenvalues of a matrix A is sometimes called the *spectrum* of A , and the following description of the eigenvalues is called a *spectral theorem*.

Theorem 3

The Spectral Theorem for Symmetric Matrices

An $n \times n$ symmetric matrix A has the following properties:

- A has n real eigenvalues, counting multiplicities.
- The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.
- The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.
- A is orthogonally diagonalizable.

Spectral Decomposition

Suppose $A = PDP^{-1}$, where the columns of P are orthonormal eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ of A and the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ are in the diagonal matrix D . Then, since $P^{-1} = P^T$,

$$\begin{aligned}
 \text{for } n=3: \quad \vec{u}_1 &= \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{1z} \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} u_{2x} \\ u_{2y} \\ u_{2z} \end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix} u_{3x} \\ u_{3y} \\ u_{3z} \end{bmatrix} \\
 A = PDP^T &= \begin{bmatrix} u_{1x} & u_{2x} & u_{3x} \\ u_{1y} & u_{2y} & u_{3y} \\ u_{1z} & u_{2z} & u_{3z} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} u_{1x} & u_{1y} & u_{1z} \\ u_{2x} & u_{2y} & u_{2z} \\ u_{3x} & u_{3y} & u_{3z} \end{bmatrix} \\
 &= \begin{bmatrix} \lambda_1 u_{1x} & \lambda_2 u_{2x} & \lambda_3 u_{3x} \\ \lambda_1 u_{1y} & \lambda_2 u_{2y} & \lambda_3 u_{3y} \\ \lambda_1 u_{1z} & \lambda_2 u_{2z} & \lambda_3 u_{3z} \end{bmatrix} \begin{bmatrix} u_{1x} & u_{1y} & u_{1z} \\ u_{2x} & u_{2y} & u_{2z} \\ u_{3x} & u_{3y} & u_{3z} \end{bmatrix} \\
 &= \begin{bmatrix} \lambda_1 u_{1x} \\ \lambda_1 u_{1y} \\ \lambda_1 u_{1z} \end{bmatrix} \begin{bmatrix} u_{1x} & u_{1y} & u_{1z} \end{bmatrix} + \begin{bmatrix} \lambda_2 u_{2x} \\ \lambda_2 u_{2y} \\ \lambda_2 u_{2z} \end{bmatrix} \begin{bmatrix} u_{2x} & u_{2y} & u_{2z} \end{bmatrix} + \begin{bmatrix} \lambda_3 u_{3x} \\ \lambda_3 u_{3y} \\ \lambda_3 u_{3z} \end{bmatrix} \begin{bmatrix} u_{3x} & u_{3y} & u_{3z} \end{bmatrix} \\
 &\Rightarrow \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T + \lambda_3 \vec{u}_3 \vec{u}_3^T
 \end{aligned}$$

This is called a spectral decomposition of A because it breaks up A into pieces determined by the spectrum (eigenvalues) of A .

Fact

Each term $\lambda_i \vec{u}_i \vec{u}_i^T$ is an $n \times n$ matrix of rank 1

Each matrix $\vec{u}_i \vec{u}_i^T$ is a projection matrix,

i.e. for each $\vec{x}$ in $\mathbb{R}^n$,

$$(\vec{u}_i \vec{u}_i^T) \vec{x}$$

is the orthogonal projection of $\vec{x}$ onto $\vec{u}_i$

EXAMPLE 4 Construct a spectral decomposition of the matrix A that has the orthogonal diagonalization

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

SOLUTION Denote the columns of P by $\mathbf{u}_1$ and $\mathbf{u}_2$. Then

$$A = 8\mathbf{u}_1\mathbf{u}_1^T + 3\mathbf{u}_2\mathbf{u}_2^T$$

To verify this decomposition of A , compute

$$\mathbf{u}_1\mathbf{u}_1^T = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}$$

$$\mathbf{u}_2\mathbf{u}_2^T = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}$$

and

$$8\mathbf{u}_1\mathbf{u}_1^T + 3\mathbf{u}_2\mathbf{u}_2^T = \begin{bmatrix} 32/5 & 16/5 \\ 16/5 & 8/5 \end{bmatrix} + \begin{bmatrix} 3/5 & -6/5 \\ -6/5 & 12/5 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = A \quad \blacksquare$$

Practice Problems

1. Show that if A is a symmetric matrix, then A^2 is symmetric.
2. Show that if A is orthogonally diagonalizable, then so is A^2 .

Solutions to Practice Problems

1. $(A^2)^T = (AA)^T = A^T A^T$, by a property of transposes. By hypothesis, $A^T = A$. So $(A^2)^T = AA = A^2$, which shows that A^2 is symmetric.
2. If A is orthogonally diagonalizable, then A is symmetric, by Theorem 2. By Practice Problem 1, A^2 is symmetric and hence is orthogonally diagonalizable (Theorem 2).

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