Sec 6.1 Review

Ex: The <u>inner product</u> of vectors $a = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and $b = \begin{bmatrix}0\\-2\\1\\1\end{bmatrix}$ is

$$\underbrace{a \cdot b}_{\text{notation}} \overset{\text{def}}{=} a^T b = [1\ 1\ 1\ 1]\begin{bmatrix}0\\-2\\1\\1\end{bmatrix} = 1(0) + 1(-2) + 1(1) + 1(1) = -2+1+1 = \boxed{0}$$

- The <u>length</u> (or <u>norm</u>) of a vector $v = \begin{bmatrix}0\\-2\\1\\1\end{bmatrix}$ is

$$\underbrace{\|v\|}_{\text{notation}} \overset{\text{def}}{=} \sqrt{v \cdot v} = \sqrt{0^2 + (-2)^2 + 1^2 + 1^2} = \boxed{\sqrt{6}}$$

Some properties: For all $\vec{v}$ in $\mathbb{R}^n$ and c in $\mathbb{R}$, we have ...

(1) $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$ (2) $\|c\vec{v}\| = |c|\ \|\vec{v}\|$

<u>Def</u>. A vector whose length is 1 is called a <u>unit vector</u>

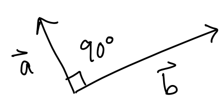
Ex: Find a unit vector u in the same direction as $v = \begin{bmatrix}0\\-2\\1\\1\end{bmatrix}$

(This process is called "normalizing $\vec{v}$")

Sol: $u = \dfrac{1}{\|v\|} v = \dfrac{1}{\sqrt{6}}\begin{bmatrix}0\\-2\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-2/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$

<u>Def</u> Two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^n$ are said to be <u>orthogonal</u> to each other if $\vec{a} \cdot \vec{b} = 0$.

In $\mathbb{R}^2$ and $\mathbb{R}^3$, two vectors $\vec{a}$ and $\vec{b}$ are perpendicular (or orthogonal) to each other if $\vec{a} \cdot \vec{b} = 0$



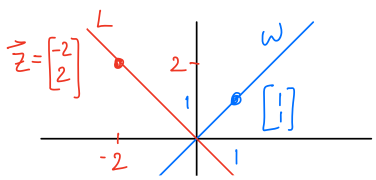
Ex: Vectors a and b in $\mathbb{R}^4$ above are orthogonal since $a \cdot b = 0$.

Def: Let W be a subspace of $\mathbb{R}^n$.

- A vector $\vec{z}$ in $\mathbb{R}^n$ is called <u>orthogonal to W</u> if $\vec{z}$ is orthogonal to every vector in W
- $W^\perp \overset{def}{=} \{$all vectors $\vec{z}$ that are orthogonal to $W\}$ (Read "W perpendicular" or "W perp") is called <u>the orthogonal complement of W</u>

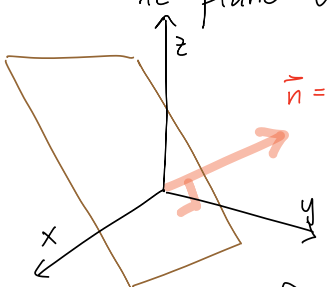
Facts:
- $W^\perp$ is also a subspace of $\mathbb{R}^n$.
- $(W^\perp)^\perp = W$

Ex: The line $W = \text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ is a subspace of $\mathbb{R}^2$.



- The vector $\vec{z} = \begin{bmatrix}-2\\2\end{bmatrix}$ is perpendicular to every vector in W, so we say "z is orthogonal to W."
- Every vector in $L = \text{Span}\left\{\begin{bmatrix}-2\\2\end{bmatrix}\right\}$ is perpendicular to each vector in W so we say the line L is $W^\perp$, the orthogonal complement of W.
- Note: $L^\perp = W$.

Ex: The plane $W = \left\{\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} : -v_1 + 2v_2 + 3v_3 = 0\right\}$ is a subspace of $\mathbb{R}^3$.



$\vec{n} = \begin{bmatrix}-1\\2\\3\end{bmatrix}$ is perpendicular to W

The line $L = \text{Span}\left\{\begin{bmatrix}-1\\2\\3\end{bmatrix}\right\}$ is the orthogonal complement $W^\perp$ and $W = L^\perp$.

Review: Sec 6.2 Orthogonal sets

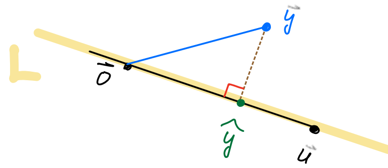
Def (Orthogonal projection of vector y onto vector u)

Given vectors y and u in $\mathbb{R}^n$, the <u>orthogonal projection of y onto u</u> $\text{proj}_u\, y$

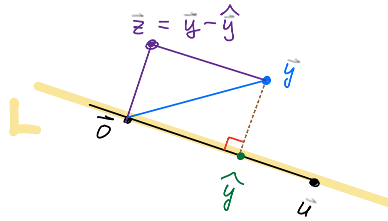
(a.k.a <u>orthogonal projection of y onto L</u> $\text{proj}_L\, y$)

where $L = \text{Span}\{u\}$ is the line containing $\vec{0}$ and u, is

$$\hat{y} = \text{proj}_u\, y = \text{proj}_L\, y \overset{\text{def}}{=} \frac{y \cdot u}{u \cdot u}\, u$$



Note: y can be written as sum $y = \hat{y} + \underbrace{(y - \hat{y})}_{\text{call this } z}$



Theorem

Suppose $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ is an <u>orthogonal set</u> of nonzero vectors in $\mathbb{R}^n$,

meaning each pair $\vec{u}_i, \vec{u}_j$ are orthogonal to each other:
$\vec{u}_i \cdot \vec{u}_j = 0$ when $i \neq j$

(1) Then S is linearly independent.

(2) Thus S is a basis for the subspace $W = \text{Span}\{S\}$

Sec 6.2 (Thm 4)

(4)

Def S is called an <u>orthogonal basis</u> for W

(3) For each $\vec{y}$ in W, the weights in the linear combination

$$\vec{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_p \vec{u}_p$$

are given by $c_1 = \dfrac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}$, $c_2 = \dfrac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}$, $\dots$, $c_p = \dfrac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p}$

Sec 6.2 (Thm 5)

Ex Let $S = \left\{ \underset{u_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}} \right\}$

a) Check that S is an orthogonal set.

Sol: $u_1 \cdot u_3 = 0$ from above

$u_2 \cdot u_3 = 3(-1) + 1(2) + 1(1) = 0$

Also check $u_1 \cdot u_3 \overset{?}{=} 0$

b) This means S is an orthogonal basis for $W_1 = \text{Span}\{S\}$

Express $y = \begin{bmatrix} 7 \\ 3 \\ -3 \\ -3 \end{bmatrix}$ as a linear combination of S.

Normally, we would need to do $\left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & \begin{matrix} 7 \\ 3 \\ -3 \\ -3 \end{matrix} \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Now we can use part (3) of the theorem above.

Sol: Compute

$$y \cdot u_1 = \begin{bmatrix} 7 \\ 3 \\ -3 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 7 + 3 - 3 - 3 = 4, \quad u_1 \cdot u_1 = 1 + 1 + 1 + 1 = 4, \quad c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1} = \frac{4}{4} = 1$$

$$y \cdot u_2 = \begin{bmatrix} 7 \\ 3 \\ -3 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = -21 + 3 - 3 - 3 = -24, \quad u_2 \cdot u_2 = 9 + 1 + 1 + 1 = 12, \quad c_2 = \frac{y \cdot u_2}{u_2 \cdot u_2} = \frac{-24}{12} = -2$$

$$y \cdot u_3 = \begin{bmatrix} 7 \\ 3 \\ -3 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} = 0 - 6 - 3 - 3 = -12, \quad u_3 \cdot u_3 = 0 + 4 + 1 + 1 = 6, \quad c_3 = \frac{y \cdot u_3}{u_3 \cdot u_3} = \frac{-12}{6} = -2$$

So $y = u_1 - 2u_2 - 2u_3$

Verify by computing the RHS

IV Orthonormal sets

Def. An orthogonal set $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ is called an orthonormal set if each element is a unit vector.

Def/Thm S is called an orthonormal basis for $W = \text{Span}\{S\}$.

Ex: The standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ is an orthonormal basis of $\mathbb{R}^n$.

Ex: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis of $W = \text{Span}\{\vec{e}_1, \vec{e}_2, \vec{e}_4\}$, a subspace of $\mathbb{R}^4$.

Thm (Thm 7) Suppose U is an $m \times n$ matrix with orthonormal columns, and let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then:

a) $\| U\vec{x} \| = \| \vec{x} \|$ (The linear transformation using U preserves length)

b) $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$

c) $(U\vec{x}) \cdot (U\vec{y}) = 0$ if and only if $\vec{x} \cdot \vec{y} = 0$ (The linear map using U preserves orthogonality)

Example 6 (Sec 6.2)

Let $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$. Notice that U has orthonormal columns and

$$U^T U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Verify that $\|U\mathbf{x}\| = \|\mathbf{x}\|$.

Sol:

$$U\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\|U\mathbf{x}\| = \sqrt{9+1+1} = \sqrt{11}$$

$$\|\mathbf{x}\| = \sqrt{2+9} = \sqrt{11}$$

Def/Theorem Let U be an $n \times n$ matrix. TFAE:

(1) U is an orthogonal matrix (possibly confusing name)

(2) $U^T U = Id$, i.e. U is invertible with $U^{-1} = U^T$

(3) The columns of U form an orthonormal set (orthogonal set of unit vectors)

Review: Sec 6.3 Orthogonal projections (onto a subspace)

The Orthogonal Decomposition Theorem (Sec 6.3 Thm 8)

Let W be a subspace of $\mathbb{R}^n$ and $\vec{y}$ be a vector in $\mathbb{R}^n$.

(a) Then we can decompose $\vec{y}$ into the sum of 2 vectors:

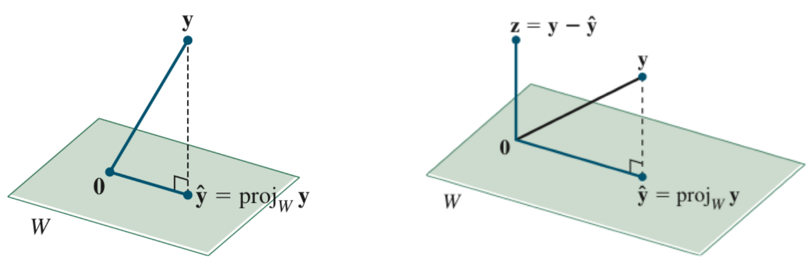
$$\vec{y} = \hat{y} + \vec{z} \qquad (*)$$

$\hat{y}$ is in W and $\underbrace{\vec{z} \text{ is orthogonal to } W}_{z \text{ is in } W^{\perp}}$

(b) If $\{u_1, \dots, u_p\}$ is an orthogonal basis of W,

then $\boxed{\hat{y} = \underbrace{\frac{y \cdot u_1}{u_1 \cdot u_1} u_1}_{\text{proj}_{u_1} y} + \dots + \underbrace{\frac{y \cdot u_p}{u_p \cdot u_p} u_p}_{\text{proj}_{u_p} y} \quad \text{and} \quad z = y - \hat{y}}$

immediate from eq (*)



Def The vector $\hat{y}$ is called <u>the orthogonal projection of y onto W</u>, denoted $\text{proj}_W y$.

The best approximation theorem (Sec 6.3 Thm 9)

Let W be a subspace of $\mathbb{R}^n$, let $\vec{y}$ be a vector in $\mathbb{R}^n$.

Then the closest point in W to $\vec{y}$ is

$\hat{y} = \text{proj}_W y$, the orthogonal projection of $\vec{y}$ onto W,

i.e.

$\| y - \hat{y} \| < \| y - v \|$ for any v in W that's not equal to $\hat{y}$.

Ex Let $u_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $u_2 = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$, $W_2 = \text{Span}\{u_1, u_2\}$, and $y = \begin{bmatrix}7\\3\\-3\\-3\end{bmatrix}$

(See also Sec 6.3 Example 2)

(a) Write y as the sum of $\underbrace{\text{a vector in } W_2}_{\hat{y} = \text{proj}_W y}$ and $\underbrace{\text{a vector orthogonal to } W}_{z = y - \hat{y}}$.

Sol: We checked earlier that $\{u_1, u_2\}$ is an orthogonal basis for W_2, so

$$\hat{y} = \text{proj}_{u_1} y + \text{proj}_{u_2} y = \underbrace{\frac{y\cdot u_1}{u_1\cdot u_1}} u_1 + \underbrace{\frac{y\cdot u_2}{u_2\cdot u_2}} u_2 = 1\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - 2\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1+6\\1-2\\1-2\\1-2\end{bmatrix} = \begin{bmatrix}7\\-1\\-1\\-1\end{bmatrix}$$

We computed earlier, copied below

$$\left[\begin{array}{l} y\cdot u_1 = 4,\quad u_1\cdot u_1 = 4,\quad \dfrac{y\cdot u_1}{u_1\cdot u_1} = \dfrac{4}{4} = 1, \text{ so } \text{proj}_{u_1} y = 1\, u_1 \\ y\cdot u_2 = -24,\quad u_2\cdot u_2 = 12,\quad \dfrac{y\cdot u_2}{u_2\cdot u_2} = \dfrac{-24}{12} = -2, \text{ so } \text{proj}_{u_2} y = -2u_2 \end{array}\right]$$

$$z = y - \hat{y} = \begin{bmatrix}7\\3\\-3\\-3\end{bmatrix} - \begin{bmatrix}7\\-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}7-7\\3+1\\-3+1\\-3+1\end{bmatrix} = \begin{bmatrix}0\\4\\-2\\-2\end{bmatrix}, \quad \text{so} \quad \boxed{\underset{y}{\begin{bmatrix}7\\3\\-3\\-3\end{bmatrix}} = \underset{\hat{y}}{\begin{bmatrix}7\\-1\\-1\\-1\end{bmatrix}} + \underset{z}{\begin{bmatrix}0\\4\\-2\\-2\end{bmatrix}}}$$

Confidence check: z should be in $W_2^{\perp}$. Check $z\cdot u_1 \overset{?}{=} 0$ and $z\cdot u_2 \overset{?}{=} 0$

$$\begin{bmatrix}0\\4\\-2\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \overset{\checkmark}{=} 0 \text{ and } \begin{bmatrix}0\\4\\-2\\-2\end{bmatrix}\cdot\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} \overset{\checkmark}{=} 0$$

(See also Sec 6.3 Example 3)

(b) What is the closest point in W_2 to y?

Sol: By the "Best Approximation Theorem",

the closest point is $\text{proj}_{W_2} y$ which is $\hat{y} = \boxed{\begin{bmatrix}7\\-1\\-1\\-1\end{bmatrix}}$

(See also Sec 6.3 Example 4)

(c) What is the distance from y to the nearest point in W_2?

Sol: The distance from y to the nearest point $\hat{y}$ is

the length $\|y - \hat{y}\| = \sqrt{0^2 + 4^2 + 2^2 + 2^2} = \boxed{\sqrt{24}}$

Practice Problems

(Sec 6.1)

1. Let $\mathbf{a} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Compute $\dfrac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}$ and $\left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a}$.

2. Let $\mathbf{c} = \begin{bmatrix} 4/3 \\ -1 \\ 2/3 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$.

   a. Find a unit vector $\mathbf{u}$ in the direction of $\mathbf{c}$.

   b. Show that $\mathbf{d}$ is orthogonal to $\mathbf{c}$.

   c. Use the results of (a) and (b) to explain why $\mathbf{d}$ must be orthogonal to the unit vector $\mathbf{u}$.

Solutions to Practice Problems

1. $\mathbf{a} \cdot \mathbf{b} = 7$, $\mathbf{a} \cdot \mathbf{a} = 5$. Hence $\dfrac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} = \dfrac{7}{5}$, and $\left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} = \dfrac{7}{5}\mathbf{a} = \begin{bmatrix} -14/5 \\ 7/5 \end{bmatrix}$.

2.

   a. Scale $\mathbf{c}$, multiplying by 3 to get $\mathbf{y} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$. Compute $\|\mathbf{y}\|^2 = 29$ and $\|\mathbf{y}\| = \sqrt{29}$.

   The unit vector in the direction of both $\mathbf{c}$ and $\mathbf{y}$ is $\mathbf{u} = \dfrac{1}{\|\mathbf{y}\|}\mathbf{y} = \begin{bmatrix} 4/\sqrt{29} \\ -3/\sqrt{29} \\ 2/\sqrt{29} \end{bmatrix}$.

   b. $\mathbf{d}$ is orthogonal to $\mathbf{c}$, because

$$\mathbf{d} \cdot \mathbf{c} = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4/3 \\ -1 \\ 2/3 \end{bmatrix} = \frac{20}{3} - 6 - \frac{2}{3} = 0$$

   c. $\mathbf{d}$ is orthogonal to $\mathbf{u}$, because $\mathbf{u}$ has the form $k\mathbf{c}$ for some k, and

$$\mathbf{d} \cdot \mathbf{u} = \mathbf{d} \cdot (k\mathbf{c}) = k(\mathbf{d} \cdot \mathbf{c}) = k(0) = 0$$

Practice Problems Sec 6.2

1. Let $\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

Solutions to Practice Problems

1. The vectors are orthogonal because

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = -2/5 + 2/5 = 0$$

They are unit vectors because

$$\|\mathbf{u}_1\|^2 = \left(-1/\sqrt{5}\right)^2 + \left(2/\sqrt{5}\right)^2 = 1/5 + 4/5 = 1$$
$$\|\mathbf{u}_2\|^2 = \left(2/\sqrt{5}\right)^2 + \left(1/\sqrt{5}\right)^2 = 4/5 + 1/5 = 1$$

In particular, the set $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent, and hence is a basis for $\mathbb{R}^2$ since there are two vectors in the set.

Practice Problems (Sec 6.3)

1. Let $\mathbf{u}_1 = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -9 \\ 1 \\ 6 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. (i) Use the fact that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal to compute $\text{proj}_W\, \mathbf{y}$. (ii) What's the closest point in W to y? (iii) Find the distance from y to W.

Solution to Practice Problems

1. (i) Compute

$$\text{proj}_W\, \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \frac{88}{66}\mathbf{u}_1 + \frac{-2}{6}\mathbf{u}_2$$

$$= \frac{4}{3}\begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \\ 6 \end{bmatrix} = \mathbf{y}$$

In this case, $\mathbf{y}$ happens to be a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, so $\mathbf{y}$ is in W. (ii) The closest point in W to $\mathbf{y}$ is $\mathbf{y}$ itself. (iii) The distance is 0.

— end —