

Review of Diagonalization Sec 5.1-5.3

Sec 5.1

Def

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ . A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to λ* .¹

EXAMPLE 1 Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by A are shown in Figure 1. In fact, $A\mathbf{v}$ is just $2\mathbf{v}$. So A only “stretches” or dilates $\mathbf{v}$. ■

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\vec{v}, \text{ so } \vec{v} \text{ is an eigenvector of } A \text{ corresponding to an eigenvalue } \lambda = 2$$

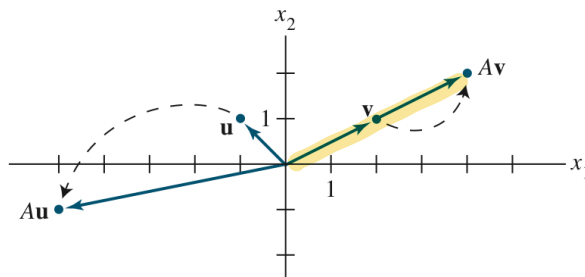


FIGURE 1 Effects of multiplication by A .

$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \text{ not a scalar multiple of } \vec{u},$$

so $\vec{u}$ is not an eigenvector of A

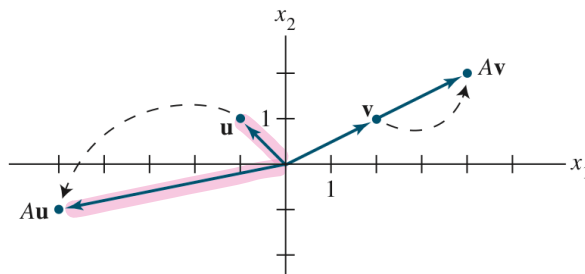


FIGURE 1 Effects of multiplication by A .

Sec 5.1

Note: $\vec{x}$ is a λ -eigenvector iff $A\vec{x} - \lambda\vec{x} = \vec{0}$
 iff $(A - \lambda Id)\vec{x} = \vec{0}$

Let A be an $n \times n$ matrix. Suppose λ is an eigenvalue

The λ -eigenspace of A is the set of solutions of $(A - \lambda Id)\vec{x} = \vec{0}$
 i.e. λ -eigenspace of A is $\text{Nul}(A - \lambda Id)$

So λ -eigenspace of A is a subspace of $\mathbb{R}^n$.

EXAMPLE 4 Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

SOLUTION Form

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for $(A - 2I)\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

At this point, it is clear that 2 is indeed an eigenvalue of A because the equation $(A - 2I)\mathbf{x} = \mathbf{0}$ has free variables. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_2 \text{ and } x_3 \text{ free}$$

The eigenspace, shown in Figure 3, is a two-dimensional subspace of $\mathbb{R}^3$. A basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

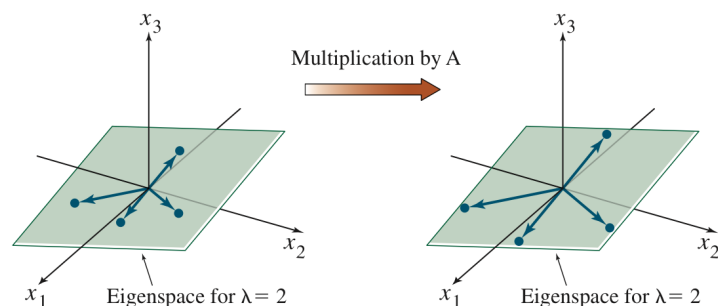


FIGURE 3 A acts as a dilation on the eigenspace.

Sec 5.2 Determinants

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Before turning to larger matrices, recall from Section 3.1 that the matrix A_{ij} is obtained from A by deleting the i th row and j th column. The determinant of an $n \times n$ matrix A can be computed by an expansion across any row or down any column. The expansion across the i th row is given by

$$\det A = (-1)^{i+1}a_{i1} \det A_{i1} + (-1)^{i+2}a_{i2} \det A_{i2} + \cdots + (-1)^{i+n}a_{in} \det A_{in}$$

The expansion down the j th column is given by

$$\det A = (-1)^{1+j}a_{1j} \det A_{1j} + (-1)^{2+j}a_{2j} \det A_{2j} + \cdots + (-1)^{n+j}a_{nj} \det A_{nj}$$

EXAMPLE 2 Compute the determinant of

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

SOLUTION Any row or column can be chosen for the expansion. For example, expanding down the first column of A results in

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{21} \det A_{21} + a_{31} \det A_{31} \\ &= 2 \det \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} - 4 \det \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + 0 \det \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix} \\ &= 2(0 - (-2)) - 4(3 - 2) + 0(-3 - 0) = 0 \end{aligned}$$

Sec 5.2

Def The characteristic equation of a matrix A is

$$\det(A - \lambda Id) = 0$$

Fact

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

Sec 5.3

A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D . The next theorem gives a characterization of diagonalizable matrices and tells how to construct a suitable factorization.

The Diagonalization Theorem (Thm 5)

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A . In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P .

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an **eigenvector basis** of $\mathbb{R}^n$.

Diagonalizing Matrices

EXAMPLE 3 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Step 1. Find the eigenvalues of A .

$$\begin{aligned} \det \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} &= (1-\lambda) \det \begin{bmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{bmatrix} - (3) \det \begin{bmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{bmatrix} + (3) \det \begin{bmatrix} -3 & -5-\lambda \\ 3 & 3 \end{bmatrix} \\ &= (1-\lambda) [(-5-\lambda)(1-\lambda) + 9] - 3 [-(5+\lambda)(1-\lambda) + 9] + (3) (-9 + 3(5+\lambda)) \\ &= -\lambda^3 - 3\lambda^2 + 4 \end{aligned}$$

$$\begin{aligned} 0 = \det(A - \lambda I) &= -\lambda^3 - 3\lambda^2 + 4 \\ &= -(\lambda - 1)(\lambda + 2)^2 \end{aligned}$$

The eigenvalues are $\lambda = 1$ and $\lambda = -2$.

Step 2. Find three linearly independent eigenvectors of A . Three vectors are needed because A is a 3×3 matrix. This is the critical step. If it fails, then Theorem 5 says that A cannot be diagonalized. The method in Section 5.1 produces a basis for each eigenspace:

$$\text{Basis for } \lambda = 1: \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Basis for } \lambda = -2: \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

You can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

Step 3. Construct P from the vectors in step 2. The vectors may be listed in any order. Using the order chosen in step 2, form

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 4. Construct D from the corresponding eigenvalues. In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P . Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to $\lambda = -2$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

It is a good idea to check that P and D really work. To avoid computing P^{-1} , simply verify that $AP = PD$. This is equivalent to $A = PDP^{-1}$ when P is invertible. (However, be sure that P is invertible!) Compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \quad \blacksquare$$

(From Sec 7.1)

EXAMPLE 3 (Orthogonally)diagonalize the matrix $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$,
given the following info:

for $\lambda = 7$: An (orthonormal) basis for the eigenspace

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

An (orthonormal) basis for the eigenspace for $\lambda = -2$ is

$$\mathbf{u}_3 = \frac{1}{\|2\mathbf{v}_3\|} 2\mathbf{v}_3 = \frac{1}{3} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

Sol Let

$$P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}, \quad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Then P (orthogonally) diagonalizes A , and $A = PDP^{-1}$. ■

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