

Sec 4.5 The dimension of a vector space

I. Def of dimension

Thm If a vector space V has a basis $B = \{b_1, b_2, \dots, b_n\}$ and
(Thm 10) S is any set in V containing more than n elements,
then S must be linearly dependent.

Contrapositive statement (equivalent):

If S is any set in V that is linearly independent,
then S contains no more than n elements.

Thm If a vector space V has a basis of n elements,
(Thm 11) then every basis of V must consist of exactly n elements.

Def: If a vector space V has a finite spanning set, then
 V is said to be finite-dimensional, and
the dimension of V is

$\dim V =$ the number of elements in a basis for V

The dimension of the zero vector space $\{ \text{zero element} \}$ is
defined to be 0.

If V doesn't have a finite spanning set, then
 V is said to be infinite-dimensional.

Ex 1. $\dim \mathbb{R}^n = n$ because the standard basis for $\mathbb{R}^n$ is $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$

$\dim \mathbb{R}^3 = 3$ because $\mathbb{R}^3$ has a basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2. $\dim \mathbb{P}_n = n+1$ because the standard basis for $\mathbb{P}_n$ is $\{1, t, t^2, \dots, t^n\}$

$\dim \mathbb{P}_2 = 3$ because $\mathbb{P}_2$ has a basis $\{1, t, t^2\}$,

so $\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$ because Thm 9 (in Sec 4.4) says every vector space of dimension n is isomorphic to $\mathbb{R}^n$

3. $\mathbb{P} = \{\text{all polynomials}\}$ is infinite-dimensional

The space of all continuous functions is infinite-dimensional.

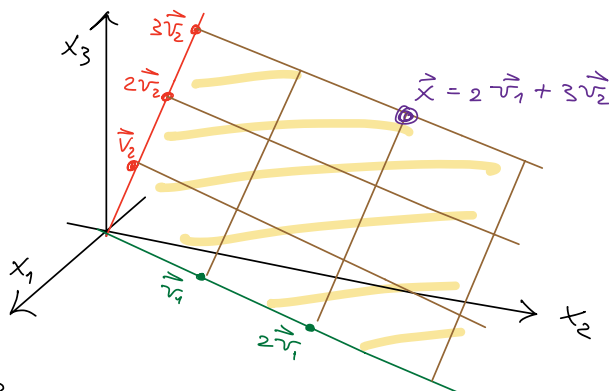
II. Subspaces of $\mathbb{R}^n$

Ex 7 from Sec 4.4 (previous lecture)

The space $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ has a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, so $\dim H = 2$

Picture

Basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ imposes a coordinate system ("graph paper") on the plane H in $\mathbb{R}^3$



Note: • H is a subspace of $\mathbb{R}^3$

• H is isomorphic to the plane $\mathbb{R}^2$ (not $\mathbb{R}^3$) because Thm 9 (in Sec 4.4) says every vector space of dimension n is isomorphic to $\mathbb{R}^n$

Fact The following are all subspaces of $\mathbb{R}^3$ (classified by dimension):

- The only 0-dimensional subspace is the zero subspace $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$
- Every 1-dimensional subspace of $\mathbb{R}^3$ is a line L through the origin.

L can be spanned by any nonzero vector contained in L .

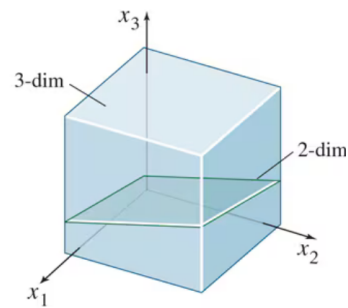
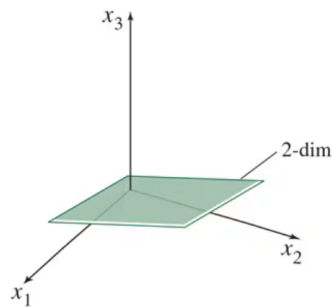
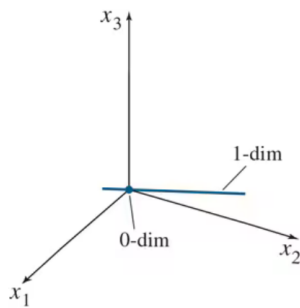
- Every 2-dimensional subspace of $\mathbb{R}^3$ is a plane H through the origin.

H can be spanned by any two linearly independent vectors in H

- The only 3-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

$\mathbb{R}^3$ can be spanned by any three linearly independent vectors in $\mathbb{R}^3$, by the Invertible Matrix Theorem

Sample subspaces of $\mathbb{R}^3$



III. The basis theorem

Thm Suppose V is a finite-dimensional vector space,
(Thm 12) and let H be a subspace of V .

- (a) Any linearly independent set S in H (if S is not already a basis for H) can be expanded to a basis for H
- (b) $\dim H \leq \dim V$

Remark: To be a basis for a vector space V , a set S must be:

- a spanning set of V , AND
- linearly independent (usually easier to verify than spanning)

BUT, if you know the dimension of V ,
and you know S has the correct number of elements,
it's enough to show one of the two properties.

Thm (The Basis Theorem)

(Thm 13) Let V be a vector space with $\dim V = p \geq 1$.

- (a) If S is a linearly independent set of p elements in V ,
then S is automatically a basis for V .
- (b) If S is a set of p elements such that $V = \text{span}\{S\}$,
then S is automatically a basis for V .

IV. The dimensions of $\text{Nul } A$ & $\text{Col } A$ (and $\text{Row } A$)

Def Let A be an $m \times n$ matrix.
rows cols

- The rank of A is $\dim \text{Col } A$ (notation: $\text{rank } A$)
- The nullity of A is $\dim \text{Nul } A$ (notation: $\text{nullity } A$)

Note $\text{rank } A = \text{number of pivot columns of } A$
 $= \dim \text{Row } A$

Why? • Our algorithm for finding a basis for $\text{Col } A$ is to take $B = \{\text{pivot columns of the original matrix } A\}$
(See Sec 4.3 pg 225-227)

- Our algorithm for finding a basis for $\text{Row } A$ is to take $B = \{\text{nonzero rows of a row echelon form of } A\}$, and each nonzero row corresponds to a pivot position
(See Sec 4.3 pg 227)

Note $\text{nullity } A = \text{number of free variables}$

$= \text{number of columns of } A \text{ that are not pivot columns}$

Why? The algorithm for finding a basis for $\text{Nul } A$ is to find a vector for each free variable.
(See Sec 4.2 pg 213-214)

Thm (The Rank Theorem)

(Thm 14) $\text{rank } A + \text{nullity } A = \text{number of columns in } A$

Proof.

$$\begin{aligned} \text{rank } A + \text{nullity } A &= \left(\begin{array}{c} \# \text{ of pivot} \\ \text{columns} \end{array} \right) + \left(\begin{array}{c} \# \text{ of non-pivot} \\ \text{columns} \end{array} \right) \text{ by above notes} \\ &= \# \text{ of columns} \\ &\quad \square \end{aligned}$$

Ex Let $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$.

Row reduce A to an echelon form:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Two pivot columns Three nonpivot columns

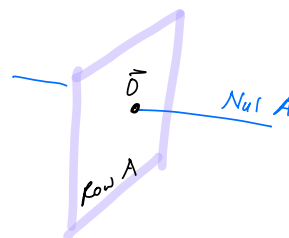
- Then:
- 1) $\dim \text{Col } A = 2 = \dim \text{Row } A$
 - 2) $\text{rank } A = 2$
 - 3) $\dim \text{Nul } A = 3$
 - 4) $\text{Nullity } A = 3$

Check: $\text{rank } A + \text{nullity } A = \text{number of columns of } A$

2 3 5

Looking ahead: We'll see later that $\text{Row } A$ and $\text{Nul } A$ are perpendicular to each other.

Cartoon in $\mathbb{R}^3$:



V. The Invertible Matrix Theorem (Continued)

Let A be an $n \times n$ matrix. TFAE:

- (1) A is invertible
- (2) The columns of A form a basis of $\mathbb{R}^n$
- (3) $\text{Col } A = \mathbb{R}^n$
- (4) $\text{rank } A = n$
- (5) $\text{nullity } A = 0$ (the number zero)
- (6) $\text{Nul } A = \{\vec{0}\}$ (the zero vector space)
 $\vec{0}$ is the zero vector $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^n$

Practice problems at the end of class

True or false? Give a reason (in 1-2 sentences)

Here V is a nonzero finite-dimensional vector space

(1) If $\dim V = 4$ and if S is a linearly dependent subset of V , then S contains more than 4 elements

Answer: False. Counter example: Let $V = \mathbb{R}^4$ and let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} \right\}$.

Here S is linearly dependent and S has no more than 4 elements.

(2) If S spans V and if T is a subset of V that contains more elements than S , then T is linearly dependent.

Answer: True. Since S is a spanning set for V , S contains a basis $\mathcal{B}$ for V (by "The Spanning Set Theorem" part (b) Thm 5 of Sec 4.3 pg 225). Since T has more elements than S , T also has more elements than $\mathcal{B}$, so (by Thm 10 of Sec 4.5 pg 241) T is linearly dependent.

(3)

Example 6

a. If A is a 7×9 matrix with nullity 2, what is the rank of A ?

b. Could a 6×9 matrix have nullity 2?

Solution

a. Since A has 9 columns, $(\text{rank } A) + 2 = 9$, and hence $\text{rank } A = 7$.

b. No. If a 6×9 matrix, call it B , had a two-dimensional null space, it would have to have rank 7, by the Rank Theorem. But the columns of B are vectors in $\mathbb{R}^6$, and so the dimension of $\text{Col } B$ cannot exceed 6; that is, $\text{rank } B$ cannot exceed 6.

—end of PDF—

(From Ex 6) # of cols of A

(a) If A is a 7×9 matrix w/ nullity 2,
what is rank A ?

Ans: rank A + nullity $A = 9$ ^{by The Rank Theorem}
rank A + 2 = 9
rank $A = \boxed{7}$

(b) Is it possible for a 6×9 matrix to have nullity 2?

Ans No. If A is such a matrix,

then its rank is 7 by the Rank Theorem.
Since rank $A = 7$, the # of pivot positions in A is 7.

But the # pivot positions in A cannot exceed 6, the # of rows.

— the end —