Sec 4.4 Coordinate systems

Idea :

- $\mathbb{R}^n$ w/ the standard basis $\mathcal{E}=\{\vec{e}_1,\dots,\vec{e}_n\}$ is nice
- Put a "Coordinate system" on any vector space V so that V acts like $\mathbb{R}^n$.

I. Def of coordinates relative to a basis $\mathcal{B}$

Thm 8 (Unique Representation Theorem)

- Let $\mathcal{B}=\{b_1, b_2, \dots, b_n\}$ be a basis for a vector space V.
- Then every element in V can be written as a linear combination of $\mathcal{B}$ in exactly one way. I.e, for every element x in V, there exist a unique set of scalars $c_1, c_2, \cdots, c_n$ such that

$$x = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n.$$

Def
- These weights $c_1, c_2, \cdots, c_n$ are called the coordinates of x relative to the basis $\mathcal{B}$ or the $\mathcal{B}$-coordinates of x.
- The vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ in $\mathbb{R}^n$, denoted by $[x]_{\mathcal{B}}$, is called the coordinate vector of x relative to $\mathcal{B}$ or the $\mathcal{B}$-coordinate vector of x.

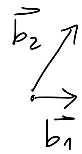
Ex $V=\mathbb{P}_2$, with basis $\mathcal{B} = \{ \overset{b_1}{1+t},\ \overset{b_2}{1+t^2},\ \overset{b_3}{t+t^2} \}$. Suppose p in V has coordinate vector (relative to $\mathcal{B}$) $[p]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$. Find p.

Sec 4.4 Practice Problem #2

Ans: The $\mathcal{B}$-coordinates of p are $5, 1, -2$, so

$$\begin{aligned} p &= 5\overbrace{(1+t)}^{b_1} + 1\overbrace{(1+t^2)}^{b_2} + (-2)\overbrace{(t+t^2)}^{b_3} \\ &= 5+5t \quad + \quad 1 + t^2 - 2t - 2t^2 \\ &= 6 + 3t - t^2 \end{aligned}$$

Ex $V = \mathbb{R}^2$, with basis $\mathcal{B} = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \right\}$.



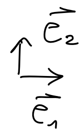
Suppose $\vec{x}$ in V has coordinate vector (relative to $\mathcal{B}$)

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\vec{x}$.

Ans: The $\mathcal{B}$-coordinates of $\vec{x}$ are -2 and 3, so

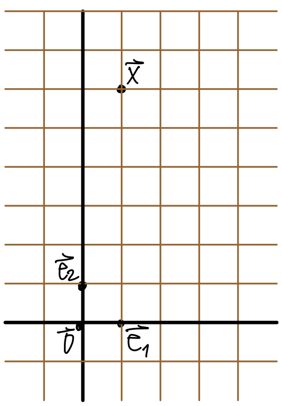
$$\vec{x} = -2\,\vec{b}_1 + 3\,\vec{b}_2 = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Ex $V = \mathbb{R}^2$, with standard basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$, $\vec{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

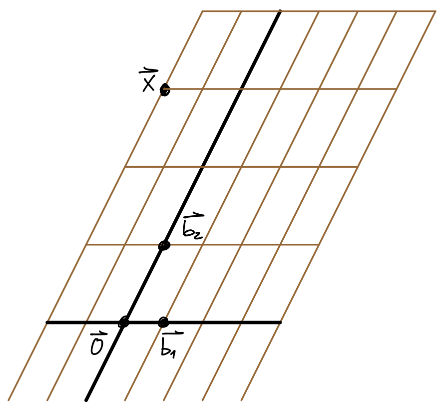


Then $\vec{x} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so the coordinate vector of $\vec{x}$

relative to the standard basis is $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$, the same as x itself



Standard graph paper for standard basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$



$\mathcal{B}$-graph paper for $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

II. How to find $\mathcal{B}$-coordinate vector of an element in $\mathbb{R}^n$

Ex Let $\mathcal{B} = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \right\}$. This is a basis for $\mathbb{R}^2$.

Find $[\vec{x}]_{\mathcal{B}}$, the coordinate vector of $\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ relative to $\mathcal{B}$.

Sol: The $\mathcal{B}$-coordinates c_1, c_2 of $\vec{x}$ are solutions to

$$c_1 \vec{b}_1 + c_2 \vec{b}_2 = \vec{x}$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \text{OR} \quad \begin{array}{cc} \vec{b}_1 & \vec{b}_2 \end{array} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

This matrix $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ (or just $P_{\mathcal{B}}$) is called the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ of $\mathbb{R}^2$

$$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ 1 & 1 & 5 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{l} c_1 = 3 \\ c_2 = 2 \end{array}$$

OR

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{4+5}{3} \\ \frac{-4+10}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

So $\vec{x} = 3\vec{b}_1 + 2\vec{b}_2$

thus $\boxed{[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$

In general: Fix a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ for $\mathbb{R}^n$.

The change-of-coordinates matrix from $\mathcal{B}$ to the standard basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ of $\mathbb{R}^n$ is the $n \times n$ matrix $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = P_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$

Then the equation

$$\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_n \vec{b}_n$$

is equivalent to

$$\boxed{\vec{x} = P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}} \quad \text{aka} \quad \vec{x} = \underbrace{\begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix}}_{P_{\mathcal{B}}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{[\vec{x}]_{\mathcal{B}}}$$

To solve for $[\vec{x}]_{\mathcal{B}}$

possible way #1:

$$\left[\begin{array}{c|c} P_{\mathcal{B}} & \vec{x} \end{array}\right] \xrightarrow{\text{Row reduce}} \left[\begin{array}{ccc|c} 1 & & & c_1 \\ & 1 & & c_2 \\ & & \ddots & \vdots \\ & & 1 & c_n \end{array}\right]$$

Augmented matrix

possible way #2:

Since the columns of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^n$, $P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Thm)

Multiplying (both sides) on the left by $P_{\mathcal{B}}^{-1}$:

$$P_{\mathcal{B}}^{-1} \vec{x} = P_{\mathcal{B}}^{-1} P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}$$

$$\boxed{P_{\mathcal{B}}^{-1} \vec{x} = [\vec{x}]_{\mathcal{B}}} \quad \text{aka} \quad \begin{bmatrix} c_1 \\ \vdots \\ c_2 \end{bmatrix} = P_{\mathcal{B}}^{-1} \vec{x}$$

III. Coordinate mapping / isomorphism of vector spaces

Fix a basis $\mathcal{B} = \{ \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \}$ for a vector space V.

<u>Def</u> The map $cm_{\mathcal{B}} : V \to \mathbb{R}^n$

$$\vec{x} \mapsto \underbrace{P_{\mathcal{B}}^{-1} \vec{x}}_{} \text{ aka. } [\vec{x}]_{\mathcal{B}}$$

is called <u>the coordinate mapping (determined by $\mathcal{B}$)</u>

<u>Fact</u> $cm_{\mathcal{B}}$ is a one-to-one and onto linear map.

<u>Proof</u> Since the standard matrix of this map is $P_{\mathcal{B}}^{-1}$ which is an invertible $n \times n$ matrix, it follows from the IMT that $cm_{\mathcal{B}}$ is one-to-one and onto.

<u>Ex (again)</u> $V = \mathbb{P}_2$, with basis $\mathcal{B} = \{ 1+t, 1+t^2, t+t^2 \}$

The coordinate mapping $cm_{\mathcal{B}} : V \to \mathbb{R}^3$

$$p \mapsto [p]_{\mathcal{B}}$$

is a linear map which is one-to-one and onto.

<u>Def</u> A linear map $T: V \to W$ between vector spaces is called a (vector space) <u>isomorphism</u> if T is one-to-one and onto.

<u>Note:</u> If there is an isomorphism from V onto W, (even if the notation and terminology for V and W may differ) V and W are indistinguishable as vector spaces.

$\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$.

Thm 9 Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis for a vector space V. Then the coordinate mapping $x \mapsto [x]_{\mathcal{B}}$ is an isomorphism from V onto $\mathbb{R}^n$

This is a big deal! This means every vector space V is "the same" (as vector spaces) as $\mathbb{R}^n$ if V has a basis with n elements.

Ex 7 Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and let $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

- Since $\vec{v}_1$ and $\vec{v}_2$ are not scalar multiples of each other, $\vec{v}_1$ & $\vec{v}_2$ are linearly independent. So $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is a basis for H.
- By Thm 9, H is isomorphic to $\mathbb{R}^2$

(a) Now, determine whether $\vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ is in H

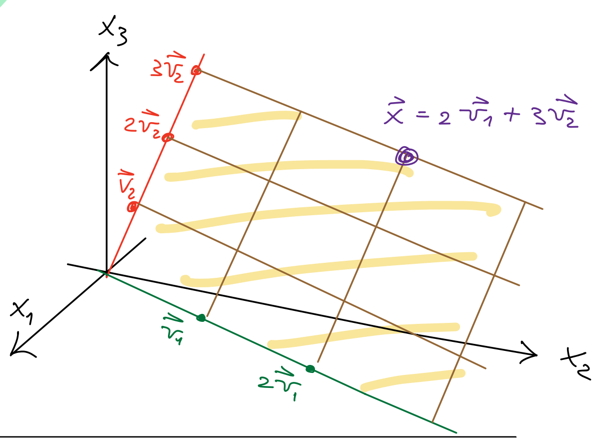
Sol: Check whether $c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{x}$ is consistent.

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} c_1 = 2 \\ c_2 = 3 \end{array}$$

So $\vec{x}$ <u>is</u> in H: $\vec{x} = 2\vec{v}_1 + 3\vec{v}_2$.

(b) Find the coordinate vector of $\vec{x}$ relative to $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$

Sol: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$



Picture

Basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ imposes a coordinate system ("graph paper") on the plane H in $\mathbb{R}^3$

IV. General change-of-coordinates matrix

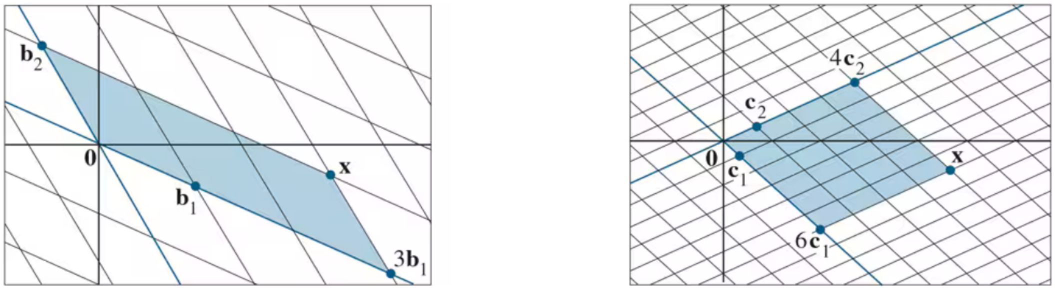
(Sec 4.6 Change of basis) Goal:

Compute <u>the change-of-coordinates matrix</u> $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P}$

<u>from a basis $\mathcal{B}$ of V to another basis $\mathcal{D}$ of V</u>

Two coordinate systems for the same vector space

Ex 1



<u>Ex 1</u> Suppose $\mathcal{B}=\{b_1, b_2\}$ and $\mathcal{D}=\{d_1, d_2\}$ are bases for a vector space V

and we know $b_1 = 4d_1 + d_2$ and $b_2 = -6d_1 + d_2$ $(**)$

Suppose $x \overset{(*)}{=} 3b_1 + b_2$. Find $[x]_{\mathcal{D}}$

the $\mathcal{D}$-coordinate vector of x

<u>Sol</u>: The equation $(*)$ tells us $[x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Apply the $\mathcal{D}$-coordinate mapping to equation $(*)$:

$$cm_{\mathcal{D}}(x) = cm_{\mathcal{D}}(3b_1 + b_2) = 3\,cm_{\mathcal{D}}(b_1) + cm_{\mathcal{D}}(b_2) \quad \text{since } cm_{\mathcal{D}} \text{ is linear}$$

$$= 3[b_1]_{\mathcal{D}} + [b_2]_{\mathcal{D}}$$

So $[x]_{\mathcal{D}} = 3[b_1]_{\mathcal{D}} + [b_2]_{\mathcal{D}}$

These three are vectors in $\mathbb{R}^2$, so we can rewrite as a matrix equation:

$$[x]_{\mathcal{D}} = \begin{bmatrix} [b_1]_{\mathcal{D}} & [b_2]_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We know $[b_1]_{\mathcal{D}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $[b_2]_{\mathcal{D}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$ by $(**)$, so $[x]_{\mathcal{D}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

Rem. $\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} [b_1]_{\mathcal{D}} & [b_2]_{\mathcal{D}} \end{bmatrix}$ is the change-of-coordinates matrix $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P}$

from $\mathcal{B}$ to $\mathcal{D}$ because we multiply it by $[x]_{\mathcal{B}}$ to get $[x]_{\mathcal{D}}$.

Thm 15 Let $\mathcal{B} = \{b_1, \dots, b_n\}$ and $\mathcal{D} = \{d_1, \dots, d_n\}$ be bases of a vector space V.

Then the matrix $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [b_1]_{\mathcal{D}} & [b_2]_{\mathcal{D}} & \cdots & [b_n]_{\mathcal{D}} \end{bmatrix}$

(called the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{D}$)

is the unique matrix satisfying

$$[x]_{\mathcal{D}} = \underset{\mathcal{D}\leftarrow\mathcal{B}}{P} \, [x]_{\mathcal{B}} \text{ for all } x \text{ in } V.$$

Fact The inverse matrix $\left[\underset{\mathcal{D}\leftarrow\mathcal{B}}{P}\right]^{-1}$ is $\underset{\mathcal{B}\leftarrow\mathcal{D}}{P}$

Why?

$$[x]_{\mathcal{D}} = \underset{\mathcal{D}\leftarrow\mathcal{B}}{P} \, [x]_{\mathcal{B}}$$

Multiply both sides by $\left(\underset{\mathcal{D}\leftarrow\mathcal{B}}{P}\right)^{-1}$

$$\left[\underset{\mathcal{D}\leftarrow\mathcal{B}}{P}\right]^{-1} [x]_{\mathcal{D}} = [x]_{\mathcal{B}}$$

input: $\mathcal{D}$-coordinate vector of x

output: $\mathcal{B}$-coordinate vector of x

IV. Computation for change of basis in $\mathbb{R}^n$

Ex 2 Let $\vec{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, and $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$

$\vec{d}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\vec{d}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, and $\mathcal{D} = \{\vec{d}_1, \vec{d}_2\}$

Find the change-of-coordinates matrix $\underset{\mathcal{D} \leftarrow \mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{D}$.

Sol: The Main Thm says

$$\underset{\mathcal{D} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [b_1]_{\mathcal{D}} & [b_2]_{\mathcal{D}} \end{bmatrix}$$

To find $[b_1]_{\mathcal{D}}$, we need to solve

$$b_1 = x_1 d_1 + x_2 d_2$$

$$\begin{bmatrix} -9 \\ 1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} -9 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{cc|c} -1 & 3 & -9 \\ -4 & -5 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -5 \end{array}\right]$$

So $[b_1]_{\mathcal{D}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$

To find $[b_2]_{\mathcal{D}}$, we need to solve

$$b_2 = y_1 d_1 + y_2 d_2$$

$$\begin{bmatrix} -5 \\ -1 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} + y_2 \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{cc|c} -1 & 3 & -5 \\ -4 & -5 & -1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \end{array}\right]$$

So $[b_2]_{\mathcal{D}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

$$\underset{\mathcal{D} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

We can solve both systems at the same time:

$$\left[\begin{array}{cc|cc} -1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

$$\underbrace{\begin{matrix} 6 & 4 \\ -5 & -3 \end{matrix}}_{\underset{\mathcal{D} \leftarrow \mathcal{B}}{P}}$$

Upshot: The only computation we need to do is

$$\left[\begin{array}{cc|cc} \vec{d}_1 & \vec{d}_2 & \vec{b}_1 & \vec{b}_2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & \underset{\mathcal{D} \leftarrow \mathcal{B}}{P} \\ 0 & 1 & \end{array}\right]$$

(Extra)

Example 3

Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$.

a. Find the change-of-coordinates matrix from C to B.

b. Find the change-of-coordinates matrix from B to C.

Solution

a. Notice that $\underset{B \leftarrow C}{P}$ is needed rather than $\underset{C \leftarrow B}{P}$, and compute

$$\left[\begin{array}{cc:cc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{c}_1 & \mathbf{c}_2 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right]$$

So

$$\underset{B \leftarrow C}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$$

b. By part (a) and property (6) (with B and C interchanged),

$$\underset{C \leftarrow B}{P} = (\underset{B \leftarrow C}{P})^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}$$

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