Sec 4.3 Linearly independent sets ; bases

Idea: Identify subsets that span a vector space V as "efficiently" as possible.

I. Def

Def Let V be a vector space.

The set $\{v_1, \dots, v_p\}$ in V is called ...

1) <u>linearly independent</u> if:

the equation $c_1 v_1 + c_2 v_2 + \dots + c_p v_p = 0$ (zero element in V)

has <u>only</u> the trivial solution $c_1 = 0, c_2 = 0, \dots, c_p = 0$.

2) <u>linearly dependent</u> if:

there are scalars $d_1, d_2, \dots, d_p$, not all zero, such that

$$d_1 v_1 + d_2 v_2 + \dots + d_p v_p = 0 \qquad (*)$$

In this case, the equation $(*)$ is called a <u>linear dependence relation among the elements $v_1, v_2, \dots, v_p$</u>.

Ex Recall $\mathbb{P} = \{\text{polynomials in } x\}$ is a vector space (w/ polynomial addition and polynomial scalar multiplication)

Previously for Sec 4.1 Lecture, we saw that we can write x^2 as a linear combination of 1, $1+x$, and $1+2x+x^2$:

$$x^2 = 1(1) + (-2)(1+x) + (1)(1+2x+x^2)$$

So, $\underbrace{(1)}_{d_1}(1) + \underbrace{(-2)}_{d_2}(1+x) + \underbrace{(1)}_{d_3}(1+2x+x^2) + \underbrace{(-1)}_{d_4} x^2 = 0$

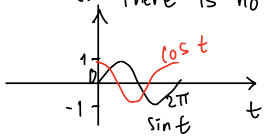
Therefore the set $\{1, 1+x, 1+2x+x^2, x^2\}$ is <u>linearly dependent</u> in $\mathbb{P}$.

Ex • The set $C[0,2\pi]$ of all continuous functions on $0 \le t \le 2\pi$ forms a vector space (w/ pointwise addition and scalar multiplication).

• The set $\{\sin t, \cos t\}$ is linearly independent in $C[0,2\pi]$ because

$\sin t$ and $\cos t$ are not multiples of one another in $C[0,2\pi]$,

i.e. there is no scalar c such that $c \cos t = \sin t$ for all t in $[0, 2\pi]$.



e.g. $\cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$

and there is no scalar such that $c\,0 = 1$.

Main Def Let H be a subspace of a vector space V (possibly $H = V$).

A set $\mathcal{B}$ of elements in V is a basis for H if ...

1) $\mathcal{B}$ is a linearly independent set, AND

2) $H = \text{Span}\, \mathcal{B}$,

i.e. every element of the subspace H can be written as a linear combination of the elements in the set $\mathcal{B}$

Ex: Let $H = \mathbb{R}^n$, and let A be an $n \times n$ matrix

The Invertible Matrix Theorem (Sec 2.3 pg 121) says:

A is invertible if and only if

the columns of A are linearly independent if and only if

the columns of A span $\mathbb{R}^n$

• So, if A is invertible, then the columns of A are

(1) linearly independent and

(2) span $\mathbb{R}^n$,

and thus the columns of A form a basis for $\mathbb{R}^n$.

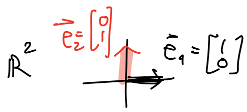
• If A is non-invertible, the columns of A

are not linearly independent and also don't span $\mathbb{R}^n$.

Def Let $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ be the columns of the $n \times n$ identity matrix,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The set $\{\vec{e}_1, \dots, \vec{e}_n\}$ is called the <u>standard basis</u> for $\mathbb{R}^n$



Def/fact The set $\{1, t, t^2, \dots, t^n\}$ is a basis for $\mathbb{P}_n$, called <u>the standard basis</u> for $\mathbb{P}_n$

Ex: The standard basis for $\mathbb{P}_2$ is $\{1, t, t^2\}$.

II. Spanning Set Thm

Idea: Given a spanning set, we can construct a basis by removing vectors we don't need

Thm Every spanning set for a vector space H contains a basis for H.

More specifically, Let $S = \{v_1, \dots, v_p\}$ be a set in a vector space V, and let $H = \text{Span}\{v_1, \dots, v_p\}$.

a.) If one of the elements in S, say, v_k, is a linear combination of the remaining elements, then the set formed by removing v_k from S is still a spanning set of H.

b.) If $H \neq \{\text{zero element}\}$, then some subset of S is a basis for H. I.e. every nonzero vector space contains a basis

III Bases for Nul A, Col A, and Row A

III(i) Find a basis for Nul A:

The method for finding the spanning set of Nul A in Sec 4.2 (see also the method for solving a linear system in Sec 1.5) produces a linearly independent set, and thus this spanning set is a basis for Nul A

Ex: Find a basis for Nul A, where $A=\begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix}$

Write the augmented matrix of the system $A\vec{x}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 2 & -4 & 1 & 0 & 0 \\ 1 & -2 & 2 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is the reduced echelon form

free variables

The general solution is $x_1 - 2x_2 + x_4 = 0$

$x_3 - 2x_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

the non-leading variables are the free variables

$\boxed{\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\} \text{ is a basis for Nul } A.}$ $\square$

III (ii) Find a basis for Col A:

Step 1: Put A into row echelon form B (doesn't need to be reduced)

Step 2: Check which columns of B have pivot positions.

Step 3: Keep **only** the columns of A which are pivot columns.

The result is a basis for Col A.

Take the original columns from A, not B!

Ex Find a basis for Col A, where $A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix}$ as before

$$A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix} \xrightarrow{\text{Row reduce}} B = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in a (row) echelon form that's not reduced

pivot positions in columns 1 and 3

A basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$

III (iii) Find a basis for Row A:

Step 1: Put A into row echelon form (doesn't need to be reduced) B

Step 2: The nonzero rows of B form a basis for Row A.

Ex: A basis for Row A for the previous example is $\{ [1 \ -2 \ -1 \ 3], [0 \ 0 \ 3 \ -6] \}$

Take the rows from B, not the original rows from A!

IV Views of a basis

A set $S = \{v_1, v_2, \dots, v_n\}$ in a subspace H is ...

- ... a <u>spanning set for H</u> if every element of H can be written as a linear combination of S in <u>at least one way</u>
  (possibly many ways, making S too big)
- ... a <u>linearly independent set</u> if every element of H can be written as a linear combination of S in <u>at most one way</u>.
  (possibly no way, making S too small to be a spanning set)
- ... a <u>basis for H</u> if every element of H can be written as a linear combination of S in <u>exactly one way</u>.
  perfect!

<u>Two views of basis:</u>

1) A basis is a spanning set that is as small as possible.
2) A basis is a linearly independent set that is as big as possible.

<u>Ex:</u> $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix} \right\}$ spans $\mathbb{R}^3$ but is not linearly independent, so not a basis

$\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$

$\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix} \right\}$ is linearly independent but does not span $\mathbb{R}^3$, so not a basis for $\mathbb{R}^3$

<u>Ex:</u> Let $H = \text{Span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix} \right\}$. Then $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix} \right\}$ is a basis for H.

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