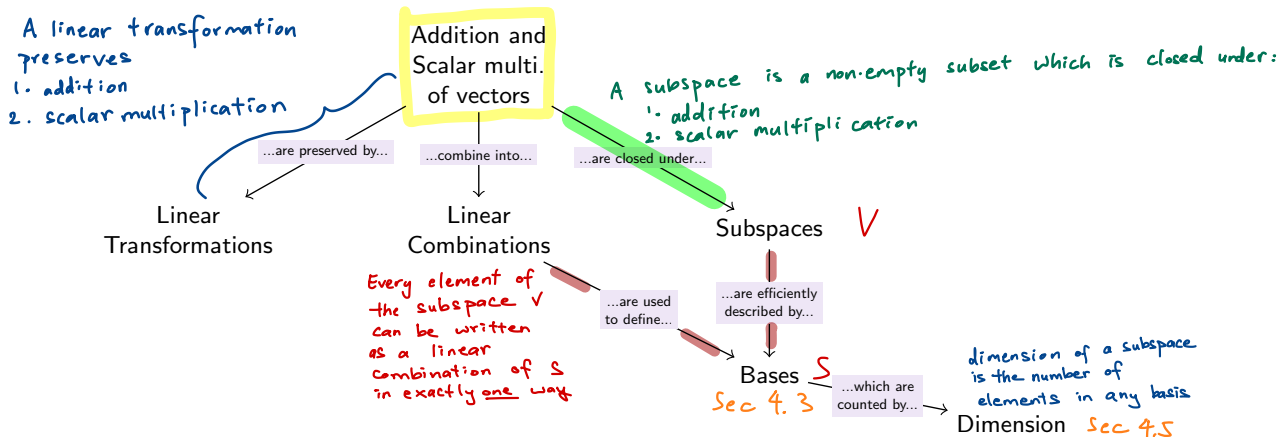


Sec 4.1 Vector spaces and subspaces

- Observation 1: Many linear algebra concepts can be defined in terms of addition and scalar multiplication



- Observation 2: Addition and scalar multiplication make sense for many other mathematical objects.

Examples

- Polynomials

polynomial addition

$$(1 + x^2) + (7 - 3x + x^3)$$

- Real-valued functions of x

$$\sin(x) + e^x$$

function addition

scalar multiplication

$$4(1 + 3x + 4x^2)$$

$$4 \ln(x)$$

scalar multiplication

4 is a scalar/number

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Vector space

Goals: Generalize what we've learned so far about vectors to other kinds of objects we can add and scalar multiply.

Definition: A vector space

A **vector space** is a set V in which

- there is a rule to **add** any two elements v, w in V , and
- there is a rule to **multiply** any v in V by any **scalar** r in $\mathbb{R}$,

such that the **axioms** on the next slide hold.

Intuitively, a vector space is a set of mathematical objects which collectively **behave like a set of vectors**.

Possibly confusing terminology

- Elements of a vector space may not be vectors in $\mathbb{R}^n$
- Some textbooks (like ours) use '**vector**' to refer to any element of a vector space.

Axioms for vector space

Axioms (essential properties) of addition

- $u + v = v + u$ for all u, v in V . (addition is commutative)
- $(u + v) + w = u + (v + w)$ for all u, v, w in V . (addition is associative)
- There is an element 0 in V , such that for all v in V ,
 $v + 0 = 0 + v = v$ (additive identity, called "0", exists)
- For each v in V , there exists $-v$ in V with
 $v + (-v) = (-v) + v = 0$ (additive inverse, denoted by "-", exists)

Axioms (essential properties) of scalar multiplication

- $r(u + v) = ru + rv$ for all u, v in V and any r in $\mathbb{R}$.
- $(r + s)v = rv + sv$ for all v in V and any r, s in $\mathbb{R}$.
- $r(sv) = (rs)v$ for all v in V and any r, s in $\mathbb{R}$.
- ~~There is an element 1 such that $1v = v$ for all v in V .~~ (Typo ϕ) $\leftarrow$ This is the number 1 in $\mathbb{R}$

An **axiom** is a fact that can't be reduced to a simpler property.

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Two examples of vector spaces: $\mathbb{R}^n$ and $\mathbb{P}$

The set of vectors of height n is a vector space!

Fact (The motivating examples of vector spaces)

For each positive integer n , the set $\mathbb{R}^n$ is a vector space.

Fact (Our first non-vector vector space)

The set of polynomials in x is a vector space, denoted $\mathbb{P}$.

Useful fact: Two polynomials are equal if and only if they have the coefficients when written in **standard form**: $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$.

► **Exercise:** Determine whether $(x - 4)^3$ is a **scalar multiple** of $x^2 + x + 1$.

Is there a scalar c in $\mathbb{R}$ such that $(x-4)^3 = c(x^2 + x + 1)$?

First, put the LHS in standard form: $(x-4)(x^2 - 8x + 16) = c(x^2 + x + 1)$

$$x^3 - 12x^2 + (32 + 16)x + 64 = c(x^2 + x + 1)$$

Since $x^2 + x + 1$ has no x^3 term, this is impossible.

$\therefore (x-4)^3$ is not a scalar multiple of $x^2 + x + 1$.

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Exercise

- Write x^2 as a **linear combination** of 1, $1+x$, and $1+2x+x^2$.

(Note: Numbers like 0, 1, and 7 count as **constant** polynomials!)

We want to find a, b, c in $\mathbb{R}$ such that

$$x^2 = a \cdot 1 + b(1+x) + c(1+2x+x^2).$$

Note: This is a polynomial in one variable, x .

The letters a, b, c are just numbers we're trying to find.

Put the RHS into standard form, so that it's easy to compare the two sides.

$$x^2 = \underline{\quad} x^2 + \underline{\quad} x + \underline{\quad} 1$$

Think of $1 = x^0$

We collect all terms with x^2 , all terms with x , and all constant terms.

$$x^2 = \underline{c} x^2 + \underline{(b+2c)} x + \underline{(a+b+c)} 1$$

The only way the LHS equals RHS is if all coefficients match.

$$1 \cdot x^2 + 0x + 0 \cdot 1 = c x^2 + (b+2c)x + (a+b+c) \cdot 1$$

This tells us
$$\begin{cases} 1 = c \\ 0 = b+2c \\ 0 = a+b+c \end{cases}$$

(a system of linear equations!)

$$\begin{aligned} c &= 1 \\ b+2c &= 0 \Rightarrow b+2=0 \Rightarrow b=-2 \\ a+b+c &= 0 \Rightarrow a+(-2)+1=0 \Rightarrow a-1=0 \Rightarrow a=1 \end{aligned}$$

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$$\therefore x^2 = \underbrace{1 \cdot (1)}_a + \underbrace{(-2) \cdot (1+x)}_b + \underbrace{(1) \cdot (1+2x+x^2)}_c$$

check:

$$1 - 2(1+x) + 1(1+2x+x^2) \stackrel{?}{=} x^2 \checkmark$$

Definition: The degree of a polynomial

The **degree** of a non-zero polynomial in x is the largest power of x with non-zero coefficient.

We define $\deg(0) := -\infty$, mostly to avoid an extra case.

Fact (Polynomials of degree at most n)

For each positive integer n , the set of polynomials in x of degree at most n is a vector space, denoted $\mathbb{P}_n$.

Examples:

- $\mathbb{P}_1$ consists of polynomials $ax + b$, for a, b in $\mathbb{R}$.
- The three polynomials $(x-1)^3$, $x^2 + 3x$, and 2 are in $\mathbb{P}_3$, but the polynomials x^4 and $x^8 - 2x^3$ are not.
- $\mathbb{P}_0$ is just the constant polynomials like 0, 1, and 7, which are the same as numbers, so $\mathbb{P}_0 = \mathbb{R}$.

A non-example: Consider the set of polynomials of degree exactly 3.

Consider $S := \{\text{polynomials of degree exactly 3}\}$.

Then $x^3 + x$ and $-x^3$ are in S , but their sum is not.

We say " S is not closed under addition."

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By a **sequence**, we mean an infinite list of real numbers.

Examples of sequences

0, 1, 1, 2, 3, 5, 8, 13, 21, ...	(the Fibonacci sequence)
2, 3, 5, 9, 11, 13, 17, ...	(prime numbers)
1, 3, 9, 27, 81, 243, ...	(powers of 3)
7, 12, -5, π , 3.5, 7, ...	(Just some random numbers)

Fact (The set of sequences is a vector space)

The set of sequences is a vector space, denote $\mathbb{S}$. Addition and scalar multiplication are defined **term-wise**.

Fact (Sets of matrices of fixed size are vector spaces)

For positive integers m and n , the set of $m \times n$ -matrices is a vector space, denoted $\mathbb{R}^{m \times n}$.

Addition and scalar multiplication are the matrix versions.

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Example: Let's say we know $\mathbb{P}$ is a vector space, but not $\mathbb{P}_3$.

- ▶ To add two polynomials in $\mathbb{P}_3$, add them as polynomials in $\mathbb{P}$, and observe the result is still in $\mathbb{P}_3$.
- ▶ Scalar multiplication also does not leave $\mathbb{P}_3$.

Since the axioms hold in $\mathbb{P}$, they automatically hold in $\mathbb{P}_3$. **So $\mathbb{P}_3$ is a vector space.**

Definition: Subspace of a vector space

Let V be a vector space. A **subspace** of V is a **non-empty** subset W of V which is...

- a. closed under addition**; that is,

for all v, w in W , the sum $v + w$ is in W , and

- b. closed under scalar multiplication**; that is,

for all v in W and c in $\mathbb{R}$, the product cv is in W .

a. Show the zero element is in W

Fact (Subspaces are vector spaces)

A subspace of a vector space is also a vector space.

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Note:

Here V is

the big vector space and W is a subset of V .

Exercise

Let S denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 0$. That is,

$$S = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 0\}.$$

Show whether S is a subspace or not a subspace of $\mathbb{P}_2$.

SAMPLE
STUDENT
ANSWER

- a. The polynomial $x-5$ is in $\mathbb{P}_2$ and plugging in 5 into $x-5$ gives 0.
degree is $1 \leq 2$

So $x-5$ is in S . This shows S is nonempty.

- b. Let f and g be in S .

That is, f and g are polynomials with degree 2 or smaller, and

$$f(5) = 0$$

$$g(5) = 0.$$

So $f+g$ is a polynomial with degree 2 or smaller, and

$$\begin{aligned}(f+g)(5) &= f(5) + g(5) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

Therefore $f+g$ is in S . So S is closed under addition.

- c. Let c be in $\mathbb{R}$ and let f be in S .

That is, f is a polynomial in x with degree 2 or smaller, and

$$f(5) = 0.$$

Then cf is also in $\mathbb{P}_2$ and

$$\begin{aligned}(cf)(5) &= c \cdot f(5) \\ &= c \cdot 0 \\ &= 0\end{aligned}$$

So cf is in S . Therefore S is closed under scalar multiplication.

Thus, S is a subspace of $\mathbb{P}_2$.

— the end —

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Exercise

Let T denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 1$. That is,

$$T = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 1\}.$$

Show whether T is a subspace or not a subspace of $\mathbb{P}_2$.

SAMPLE STUDENT ANSWER

Let $f(x) := x-4$, which is in T .

$$\begin{aligned}\text{Then } (f+f)(x) &= x-4 + x-4 \\ &= 2x-8\end{aligned}$$

$$\begin{aligned}\text{so } (f+f)(5) &= 2(5)-8 \\ &= 2\end{aligned}$$

Since $(f+f)(5) \neq 1$, $f+f$ is not in T .
So T is not closed under addition.

Therefore T is not a subspace of $\mathbb{P}_2$.

— the end —

ANOTHER SAMPLE STUDENT ANSWER

Let $f(x) := x-4$, which is in T .

$$\begin{aligned}\text{Then } (4f)(x) &= 4(x-4) \\ &= 4x-16.\end{aligned}$$

$$\begin{aligned}\text{So } (4f)(5) &= 20-16 \\ &= 4\end{aligned}$$

Since $(4f)(5) \neq 1$,

$4f$ is not in T .

So T is not closed under scalar multiplication.

Therefore T is not a subspace of $\mathbb{P}_2$.

— the end —

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III. A subspace spanned by a set

Definition: $\text{Span}\{v_1, v_2, \dots, v_p\}$

The **span** of a set of objects is the set of their linear combinations.

Example:

$$\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 0 \\ -4 \end{bmatrix} \right\} := \left\{ t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -10 \\ 0 \\ -4 \end{bmatrix} \text{ for all } t, s \text{ in } \mathbb{R} \right\}$$
$$\text{Span} \left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\} := \left\{ r \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + s \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + t \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \text{ for all } r, s, t \text{ in } \mathbb{R} \right\}$$

Theorem: Spans are subspaces

The span of a set of objects in a vector space V is a subspace of V .

- $\text{Span}\{v_1, v_2, \dots, v_p\}$ is called **the subspace** spanned by $\{v_1, v_2, \dots, v_p\}$.

Definition: Spanning sets

A **spanning set** (or **generating set**) of H is a set of objects whose span is H .

- If $H = \text{Span}\{v_1, v_2, \dots, v_p\}$, then $\{v_1, v_2, \dots, v_p\}$ is a **spanning set** for H

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Exercise

Let W be the subset of $\mathbb{R}^3$ consisting of vectors whose second entry is the average of the other two. Show that W is a subspace of $\mathbb{R}^3$.

Sol: W is the set of all vectors of the

$$\text{form } \begin{bmatrix} a \\ \frac{a+b}{2} \\ b \end{bmatrix} = \begin{bmatrix} a \\ \frac{1}{2}a + \frac{1}{2}b \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \quad \text{where } a, b \text{ are real numbers.}$$

$$\text{This implies that } W = \text{Span} \left\{ \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\},$$

so W is a subspace of $\mathbb{R}^3$.

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