

Sec 1.9 The matrix of a linear transformation

Pg 1

Big Thm: Every linear map from $\mathbb{R}^n \rightarrow \mathbb{R}^m$

can be described by a matrix transformation $\vec{x} \mapsto A\vec{x}$

I, Standard matrix for the linear transformation

- To find this matrix A , it's enough to look at the columns of the $n \times n$ identity matrix I_n

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \left\{ \begin{array}{l} \text{height} \\ n \end{array} \right\}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \vec{e}_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{ k-th position}, \quad \dots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

Ex:

- Suppose we know T is a linear transformation $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $T(\vec{e}_3) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.
- With no additional information, find a formula for the image $T(\vec{x})$ for an arbitrary $\vec{x}$ in $\mathbb{R}^3$.

Sol: Write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3$

Then we have

$$\begin{aligned} T(\vec{x}) &= T(x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3) \\ &= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + x_3 T(\vec{e}_3) \quad \text{since } T \text{ is linear} \\ &= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} x_1 + 3x_2 + 5x_3 \\ 2x_1 + 4x_2 + 6x_3 \end{bmatrix} \end{aligned}$$

Thm (Thm 10)

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map.

Then there is a unique $m \times n$ matrix A such that
meaning,
exactly one

$$T(\vec{x}) = A\vec{x} \quad \text{for all } \vec{x} \text{ in } \mathbb{R}^n.$$

This unique matrix is $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_n) \end{bmatrix}$ is
called the standard matrix for T

the concatenation of the vectors $T(\vec{e}_1), \dots, T(\vec{e}_n)$.

Ex • Suppose we know T is a linear transformation $\mathbb{R}^3 \rightarrow \mathbb{R}^2$
such that $T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $T(\vec{e}_3) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

• Find the standard matrix for T

Sol: $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

II. Geometric linear maps of $\mathbb{R}^2$

Pg 2

Rotation

The map $R: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that rotates each point in $\mathbb{R}^2$

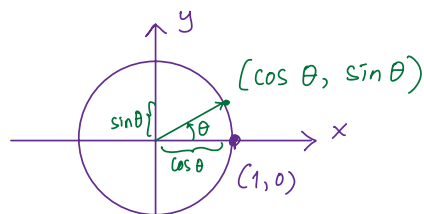
about the origin through an angle θ

(Counterclockwise for positive θ)

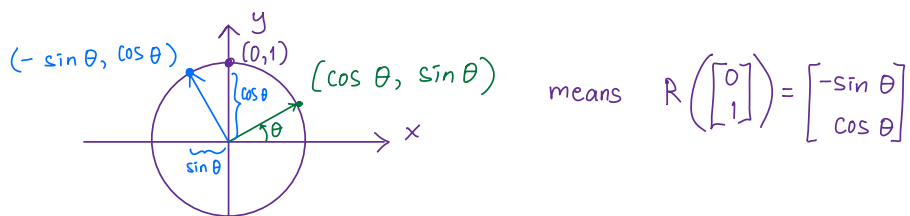
is a linear map.

What is the standard matrix A of this rotation R ?

Sol:



means $R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$



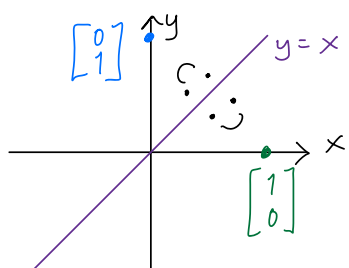
$$\text{So } A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$\text{E.g. if } \theta = \frac{\pi}{2} \text{ then } A = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Reflection

The map $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that reflects each point in $\mathbb{R}^2$ through a line l (that passes through the origin) is a linear map.

Ex: The standard matrix for the reflection through the line $y = x$ is ... $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



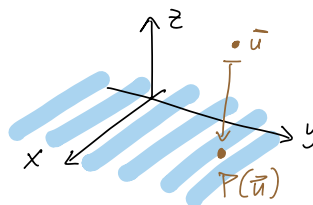
F maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Projection

The mapping $P: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

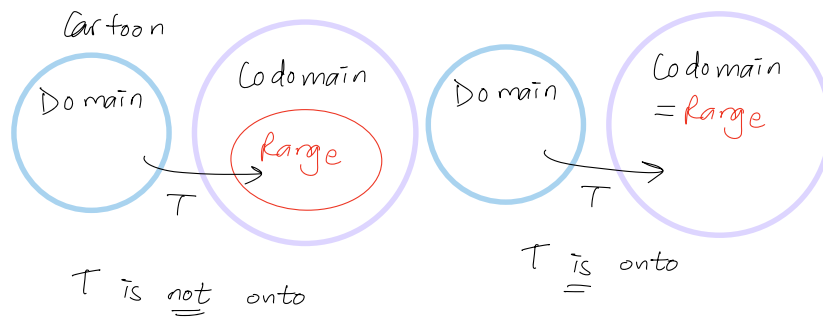
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

is called a projection because P projects points in $\mathbb{R}^3$ onto xy -plane



III. Existence & uniqueness questions: surjectivity pg 3

Def A map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called onto $\mathbb{R}^m$ or surjective if... for every $\vec{b}$ in $\mathbb{R}^m$, there is at least one $\vec{x}$ in $\mathbb{R}^n$



Ex The projection $P: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

is not onto. For example, $\vec{b} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$ is in codomain $\mathbb{R}^3$ but $\vec{b}$ is not the image $P(\vec{x})$ of any $\vec{x}$ in domain $\mathbb{R}^3$.

Ex However, the projection $P': \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is onto. For any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$,

we have $P' \left(\begin{bmatrix} x_1 \\ x_2 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

so every $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$ is in the image of P' .

In general, we can answer this existence question pg 4
by looking at an echelon form.

Thm

Let A be an $m \times n$ matrix.

Then the following are equivalent (TFAE):

I. For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$
has a solution

II. The columns of A span $\mathbb{R}^m$

III. A has a pivot in every row

IV (New in Sec 1.9) The map $\mathbb{R}^n \rightarrow \mathbb{R}^m$
 $\vec{x} \mapsto A\vec{x}$
maps onto $\mathbb{R}^m$

Thm 4
Sec 1.4
Pg 39

Ex If T is the linear transformation

whose standard matrix can be row-reduced to

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix} \quad \text{then } T \text{ is } \underline{\underline{\text{onto}}}$$

Reason: Every row has a pivot.

Ex If T is the linear transformation

whose standard matrix can be row-reduced to

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{then } T \text{ is } \underline{\underline{\text{not}}} \text{ onto}$$

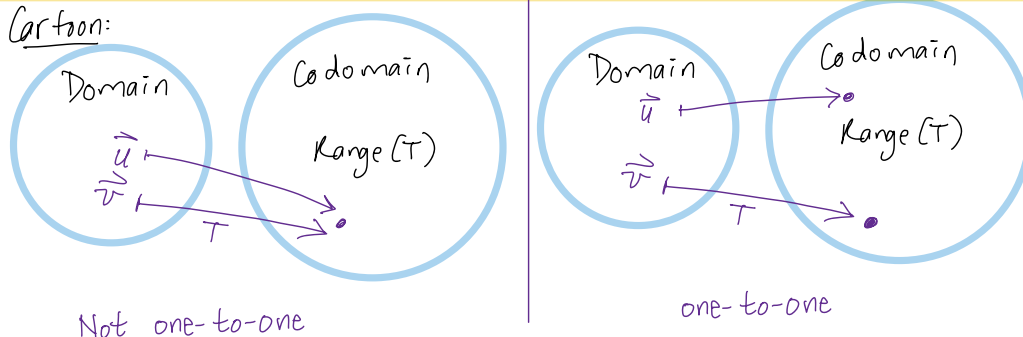
Reason: Row 3 has no pivot.

IV. Existence & uniqueness questions: injectivity

pg 5

Def A map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called one-to-one or injective if...
if $T(\vec{u}) = T(\vec{v})$ then $\vec{u} = \vec{v}$.

Cartoon:



Ex The projection $P: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

is not 1-1. For example, $P\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = P\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$

Thm Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation.

(Thm II) T is one-to-one if and only if

$T(\vec{x}) = \vec{0}$ has only the trivial solution.

Ex The projection $P': \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is also not 1-1. Reason: $P'\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So $P'(\vec{x}) = \vec{0}$ has solutions other than the trivial solution.

Thm (eq (3) in Sec 1.7, pg 61)

TFAE:

1. $A\vec{x} = \vec{0}$ has only the trivial solution
2. The columns of A are linearly independent
3. # of pivots of A = # of columns of A
4. (New Sec 1.9) The map $\vec{x} \mapsto A\vec{x}$ is one-to-one

Ex If T is the linear transformation

whose standard matrix can be row-reduced to

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix} \quad \text{then } T \text{ is } \underline{\text{not}} \text{ 1-1}$$

Reason: The augmented matrix for $A\vec{x} = \vec{0}$
can be row-reduced to

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 5 & 6 & 7 & 0 \\ 0 & 0 & 0 & 8 & 0 \end{array} \right]$$

there is one free variable for
the solution set, so $A\vec{x} = \vec{0}$
has more than just the trivial sol.

Ex If T is the linear transformation

whose standard matrix can be row-reduced to

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{then } T \text{ is also not 1-1.}$$

non-pivot columns

Reason: The solution set of $A\vec{x} = \vec{0}$ has

two free variables,

so the columns of A are not linearly independent

Ex If T is the linear transformation

whose standard matrix is $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}$

then T is 1-1.

Reason: By inspection, we see

the columns of A are linearly independent because

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ is not a scalar multiple of } \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

and vice versa.

_____ end _____