Math 2220 Notes for Ch 1 Review Handout

Consider the system of linear equations

$$\left.\begin{array}{rrrrl} x_1 & -2x_2 & -x_3 & +3x_4 & = 1 \\ 2x_1 & -4x_2 & +x_3 & & = 5 \\ x_1 & -2x_2 & +2x_3 & -3x_4 & = 4 \end{array}\right\} (*)$$

① Sec 1.3 Vector equations

Write a vector equation that is equivalent to the system $(*)$ (Sec 1.3)

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

② Sec 1.4 The matrix equation $A\vec{x} = \vec{b}$

Write a matrix equation that is equivalent to the system $(*)$ (Sec 1.4)

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} \quad \text{or} \quad A\vec{x} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

③ Sec 1.1 Systems of linear equations & Sec 1.2 Row reduction & echelon forms

④ Is the system

$$\left.\begin{array}{rrrrl} x_1 & -2x_2 & -x_3 & +3x_4 & = 1 \\ 2x_1 & -4x_2 & +x_3 & & = 5 \\ x_1 & -2x_2 & +2x_3 & -3x_4 & = 4 \end{array}\right\} (*)$$

consistent or inconsistent?

Sol:

Fact: A system of linear equations has

1. no solutions, or — the system is inconsistent
2. exactly one solution, or
3. infinitely many solutions

} the system is consistent

The augmented matrix of the system (*) is

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \qquad (**)$$

Row reduce:

$$\begin{array}{r} \\ -2\,\text{Row I} + \text{Row II} \\ -\text{Row I} + \text{Row III} \end{array} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right]$$

called pivot positions

$$\begin{array}{r} \\ \\ -\text{Row II} + \text{Row III} \end{array} \left[\begin{array}{cccc|c} \boxed{1} & -2 & -1 & 3 & 1 \\ 0 & 0 & \boxed{3} & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in a (row) echelon form

(1) All nonzero rows are above any rows of all zeros
(2) Each leading entry of a row is in a column to the right of the leading entry of the row above it
(3) All entries in a column below a leading entry are zeros

Columns 1, 3 are called pivot columns

An echelon form of the augmented matrix gives us enough information to conclude that the system is consistent

(Thm 2 in Sec 1.2)
A linear system is consistent iff an echelon form of the augmented matrix has no row of the form
$[0\ 0\ \cdots\ 0 \mid b]$ with b nonzero

Since an echelon form of the augmented matrix has no such row, the system (*) is consistent.

meaning there exist x_1, x_2, x_3, x_4 satisfying (*)

Optional: Continue to row reduce to obtain the unique reduced echelon form:

$$\tfrac{1}{3}\,\text{Row II} \quad \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(4) leading entry in each nonzero row is 1
(5) Each leading 1 is the only nonzero entry in its column

$$\text{Row II} + \text{Row I} \quad \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \checkmark$$

⑤ Sec 1.3 & 1.4

Consider the matrix $A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix}$

Is $\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$ in the subset of $\mathbb{R}^3$ spanned by the columns of A ? Why/why not ?

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ $\vec{v}_4 = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$

The subset of $\mathbb{R}^3$ spanned (or generated) by the columns of A denoted $\text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ is the collection of vectors that can be written in the form

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + c_4 \vec{v}_4 \Bigg\}$$

called linear combinations of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$

with c_1, c_2, c_3, c_4 scalars

Since system (✱) is consistent, there exist c_1, c_2, c_3, c_4 such that $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + c_4 \vec{v}_4 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$

So $\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

Thus $\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$ is in Span (columns of A).

(Answer to Q: yes!)

⑥ Sec 1.5 Solution sets of linear systems

ⓐ Describe all solutions of the nonhomogeneous system

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

$A\vec{x} = \vec{b}$ is called a <u>nonhomogeneous</u> system if $\vec{b}$ is not the zero vector

Sol:

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \xrightarrow[\text{reduce}]{\text{Row}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pivot columns

$$\begin{aligned} x_1 - 2x_2 \quad\quad + x_4 &= 2 \\ x_3 - 2x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

Basic variables

The variables that don't correspond to pivot columns, x_2, x_4 are called <u>free variables</u>.

$$\begin{cases} x_1 = 2x_2 - x_4 + 2 \\ x_2 \text{ is free} \\ x_3 = 2x_4 + 1 \\ x_4 \text{ is free} \end{cases}$$

for all choices for x_2, x_4

or

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

for all choices for x_2, x_4

ⓑ Describe all solutions of the homogeneous system

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\vec{x} = \overset{\text{zero vector}}{\vec{0}}$ is called a <u>homogeneous</u> system

Sol:

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 2 & -4 & 1 & 0 & 0 \\ 1 & -2 & 2 & -3 & 0 \end{array}\right] \xrightarrow[\text{reduce}]{\text{Row}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 \quad\quad + x_4 &= 0 \\ x_3 - 2x_4 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{cases} x_1 = 2s - t \\ x_2 = s \\ x_3 = 2t \\ x_4 = t \end{cases}$$

for all choices for s, t

or

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

for all choices for s, t

ⓒ Does $A\vec{x} = \vec{0}$ have a $\underbrace{\text{nontrivial}}_{\hookrightarrow \text{not the zero vector}}$ solution ?

Yes, e.g., set $x_2 = 3$ and $x_4 = 0$ $\vec{x} = 3\begin{bmatrix}2\\1\\0\\0\end{bmatrix} + 0\begin{bmatrix}-1\\0\\2\\1\end{bmatrix} = \begin{bmatrix}6\\3\\0\\0\end{bmatrix}$

is a nontrivial solution

Fact: A homogeneous linear system has at least one solution, $\vec{x} = \vec{0}$ called the <u>trivial solution</u>

A <u>nontrivial solution</u> is a nonzero vector $\vec{x}$ that satisfies $A\vec{x} = \vec{0}$

⑦ Sec 1.7 Linear independence

Let $\vec{v}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}-2\\-4\\-2\end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix}-1\\1\\2\end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix}3\\0\\-3\end{bmatrix}$

ⓐ Determine whether the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

<u>Fact</u>

The columns of a matrix A are linearly independent iff the matrix equation $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Since $A\vec{x} = \vec{0}$ has nontrivial solutions, the columns of A are not linearly independent (they are dependent)

ⓑ Determine by inspection whether $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

- I notice that $-2\vec{v}_1 = \vec{v}_2$, so
  $2\vec{v}_1 + \vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4 = \vec{0}$
- This means that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent by definition.

Sec 1.8 Intro to linear transformations

I. Matrix transformation

Def A <u>matrix transformation</u> is a function

$$T : \underbrace{\mathbb{R}^n}_{\text{domain}} \longrightarrow \underbrace{\mathbb{R}^m}_{\text{codomain}} \quad \text{defined by}$$

$$\underline{T(\vec{x})} = A\vec{x} \quad \text{for all } \vec{x} \text{ in } \mathbb{R}^n$$

where A is an m x n matrix (n columns, m rows)

↳ called "the image of $\vec{x}$ (under the action T)"

- <u>The range of T</u> (many textbooks refer to this set as <u>image of T</u>) is the set of all images $T(\vec{x})$.

Note: Each image $T(\vec{x})$ is of the form $A\vec{x}$,

i.e. $T(\vec{x})$ is a linear combination of the columns of A,

i.e. $T(\vec{x})$ is in the span of the columns of A.

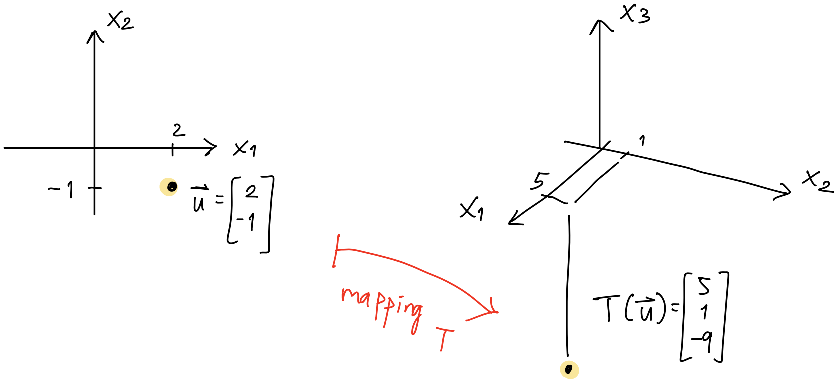
Ex 1: Define a mapping $T: \mathbb{R}^{\underset{\text{2 cols}}{2}} \longrightarrow \mathbb{R}^{\underset{\text{3 rows}}{3}}$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}}_{\text{call this 3x2 matrix } A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$$

a. Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Find the image $T(\vec{u})$ of $\vec{u}$ under the mapping T.

Sol: $$T(\vec{u}) = 2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - 1\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ -5 \\ -7 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}}$$

Sketch:



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b. Let $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ ⓘ Is $\vec{b}$ in the range of T?

(ii) How many $\vec{x}$ are there in the domain $\mathbb{R}^2$ whose image under T is $\vec{b}$?

Sol: Write the augmented matrix associated to $A\vec{x} = \vec{b}$.

$$\underbrace{\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right]}_{A \qquad \uparrow \vec{b}} \xrightarrow{\text{Row reduce}} \cdots \to \underbrace{\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{array}\right]}_{\text{(echelon form)}} \qquad \left.\begin{array}{r} x_1 - 3x_2 = 3 \\ 2x_2 = -1 \end{array}\right\}$$

- There is no row of the form $[0\,0 \mid \text{nonzero}]$ so $A\vec{x} = \vec{b}$ has at least one solution. ⓘ So $\vec{b}$ is in the range of T.
- Every column is a pivot column, so there are no free variables, meaning (ii) there is exactly one $\vec{x}$ in $\mathbb{R}^2$ whose image $T(\vec{x})$ is $\vec{b}$.

c. Let $\vec{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$. Is $\vec{c}$ in the range of T? (Extra Ex)

Sol: Write the augmented matrix associated to $A\vec{x} = \vec{c}$.

$$\underbrace{\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right]}_{A} \xrightarrow{\text{Row reduce}} \cdots \to \underbrace{\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{array}\right]}_{\text{(echelon form)}} \qquad \begin{array}{r} x_1 - 3x_2 = 3 \\ x_2 = 2 \\ 0 = -35 \end{array}$$

- The last row $[0\,0 \mid -35]$ tells us $A\vec{x} = \vec{c}$ is inconsistent. So $\vec{c}$ is <u>NOT</u> in the range of T.

II. Linear transformations

Ex of linear operations:
- Matrix transformations
- Derivatives
- Antiderivatives
- Laplace transform

Ex of non-linear operation: Quadratic function

Def Let $T : \underset{\text{domain}}{\mathbb{R}^n} \longrightarrow \underset{\text{codomain}}{\mathbb{R}^m}$ be a function.

(i) T <u>preserves vector addition</u> (or "T preserves addition")

if :

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T

(ii) T <u>preserves scalar multiplication</u>

if:

$T(c\vec{u}) = c\,T(\vec{u})$ for all $\underbrace{\text{scalars}}_{\text{numbers}}$ c and all $\vec{u}$ in the domain of T.

The mapping T is called <u>linear</u> if
T preserves addition and scalar multiplication.

Some properties of linear mappings

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map, then

1. T sends the zero vector (in $\mathbb{R}^n$) to the zero vector (in $\mathbb{R}^m$)

Why? $T(\vec{0}) = T(0\,\vec{u}) \underset{\uparrow}{=} 0\,T(\vec{u}) = \vec{0}$

by property (ii)

2. T sends linear combinations to linear combinations

Extra

$T(c\vec{u} + d\vec{v}) = c\,T(\vec{u}) + d\,T(\vec{v})$

Why? $T(c\vec{u} + d\vec{v}) \underset{\substack{\uparrow \\ \text{by property (i)}}}{=} T(c\vec{u}) + T(d\vec{v}) \underset{\substack{\uparrow \\ \text{by property (ii)}}}{=} c\,T(\vec{u}) + d\,T(\vec{v})$

Ex: Show that $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

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$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ x + 4 \\ 5y \end{bmatrix}$$

is linear or [✓ not linear]

Sol 1: I see that $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$

So it doesn't preserve scalar multiplication

Sol 2: Try $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$$T(\vec{u} + \vec{v}) = T\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -9 \\ 4 \\ 15 \end{bmatrix}$$

$$T(\vec{u}) + T(\vec{v}) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} -6 \\ 4 \\ 10 \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \\ 15 \end{bmatrix}$$

So $T(\vec{u} + \vec{v}) \neq T(\vec{u}) + T(\vec{v})$

So T doesn't preserve vector addition.

——— end of handout ———