17.8 Divergence Theorem

Application: A vector field can represent movement of liquid. The amount of water that passes across $\underbrace{\text{the barrier}}_{\text{membrane}}$ of a cell per minute can be computed using flux integral $\underbrace{\iint_S \vec{F}\cdot\vec{n}\, dS}_{\text{surface integral of a vector field}}$

Sec 17.6

Recall Green's Theorem (flux form) for $\vec{F} = \langle f, g\rangle$ think fluid flow

$$\underbrace{\oint \vec{F}\cdot\vec{n}\, ds}_{\text{flux across } C} = \iint_R \underbrace{(f_x + g_y)}_{\text{divergence}}\, dA$$

$\left(\begin{array}{l}\text{net flow across}\\ \text{boundary of } R\end{array}\right) =$ cumulative effect of the sources/sinks of the flow within R

Divergence Theorem

D : a region in $\mathbb{R}^3$, $\vec{F} = \langle f, g, h\rangle$

S : surface which is the boundary of D (normal vectors point outward)

Then $$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_D \nabla\cdot\vec{F}\, dV$$

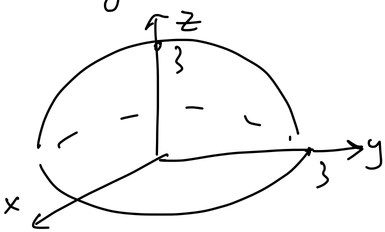
$\text{div}\,\vec{F} = f_x + g_y + h_z$

LHS: outward flux of $\vec{F}$, surface integral of a vector field (Sec 17.6)

RHS: triple integral (Sec 16.4, 16.5)

Ex 2: Let $\vec{F} = \langle \overset{f}{-y}, \overset{g}{x-z}, \overset{h}{y} \rangle$. Use Divergence Theorem to find the outward flux across the closed surface

S : the upper hemisphere $x^2+y^2+z^2=9$ for $z \geq 0$ together with its base in the xy-plane.



Sol: To compute flux $\iint_S \vec{F} \cdot \vec{n} \, dS$ directly,

we need to compute two surface integrals,

for the hemisphere and its base.

Using Divergence Theorem, we have $\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_D \operatorname{div} \vec{F} \, dV$

where D is the region in S

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = f_x + g_y + h_z = 0 + 0 + 0$$

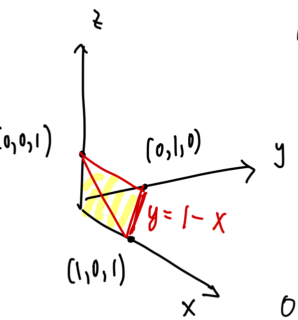
$$= \iiint_D 0 \, dV = 0$$

Flux across S is 0.

Ex (MML #4)

Use the Divergence Theorem to compute the net outward flux of $\vec{F} = \langle -x, 3y, z \rangle$ across the surface S, where S is the boundary of the tetahedron in the first octant formed by the plane $\underbrace{x + y + z = 1}_{z = 1 - x - y}$.

Sol:



Let D denote the tetahedron.

$$D = \{ (x, y, z) : 0 \le z \le 1 - x - y, \quad 0 \le y \le 1 - x, \quad 0 \le x \le 1 \}$$

$$\text{Outward flux} = \iint_S \vec{F} \cdot \vec{n} \, dS \underset{\substack{\uparrow \\ \text{Divergence} \\ \text{Theorem}}}{=} \iiint_D \operatorname{div} \vec{F} \, dV$$

$$\operatorname{div} \vec{F} = -1 + 3 + 1 = 3$$

$$= \iiint_D 3 \, dV = 3 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz \, dy \, dx$$

inner: $\displaystyle\int_0^{1-x-y} dz = (1 - x - y)$

middle: $\displaystyle\int_0^{1-x} 1 - x - y \, dy = (1-x)y - \frac{y^2}{2} \Big|_{y=0}^{y=1-x} = (1-x)^2 - \frac{(1-x)^2}{2}$

$$= \frac{(1-x)^2}{2} = \frac{1}{2}[1 - 2x + x^2]$$

outer: $\displaystyle 3 \int_0^1 \frac{1}{2}[1 - 2x + x^2] = \frac{3}{2}\left(x - \frac{2}{2}x^2 + \frac{x^3}{3} \right) \Big|_{x=0}^{x=1} = \frac{3}{2}\left(1 - 1 + \frac{1}{3}\right) = \boxed{\frac{1}{2}}$

— ended here on Monday —