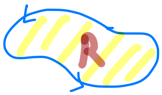
17.7 Stokes' Theorem

Recall Green's Thm connects circulation and double integral:
(circulation form)

$$\underbrace{\oint_C \langle f,g\rangle \cdot d\vec{r}}_{\text{Circulation along } C} = \iint_R \underbrace{(g_x - f_y)}_{\text{2D curl}}\, dA = \iint_R \overbrace{\left(\nabla \times \langle f,g,0\rangle\right)\cdot \hat{k}}\, dA$$

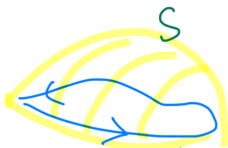
↺ pos contribution

Double integral over a region R in $\mathbb{R}^2$ (Sec 16.2-16.3)



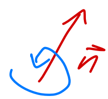
Curve C, boundary of R, Always counterclockwise (viewed from above)

Now, instead of region R in $\mathbb{R}^2$, we have surface S in $\mathbb{R}^3$.



Curve C, boundary of S

curl



positive contribution

<u>Stokes' Theorem</u> Let $\vec{F} = \langle f,g,h\rangle$ be a vector field,

S is an oriented surface (pick which direction is "up"),

C is the curve which is the boundary of S

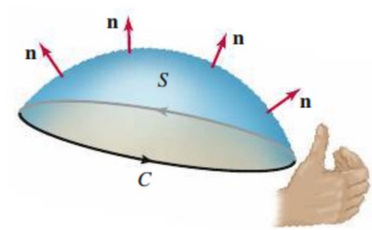
$$\left[\ \underbrace{\oint_C \vec{F}\cdot d\vec{r}}_{\text{LHS: still circulation along } C} = \underbrace{\iint_S \underbrace{(\nabla\times\vec{F})}_{\text{3D curl}}\cdot\vec{n}\, dS}_{\text{RHS: surface integral (Sec 17.6)}}\ \right]$$

Same meaning as before:
Net circulation on boundary of S = sum of "rotation" of $\vec{F}$ over S

Here $\vec{n}$ is the unit vector normal to S.
Direction of $\vec{n}$ is determined by orientation of S
(which way is "up").

S and C are related by the Right-hand rule:

Curl your right hand in the positive direction around C. Then your thumb points in the general direction of which way is "up".



The vector $\vec{n}$ points in the same direction as your thumb.

In this situation, we say that $\vec{n}$ is consistent with the orientation of C.

Recall (Sec 17.5) If $\vec{F}$ is conservative, then $\vec{F} = \nabla \varphi$ for some φ,
so $\text{curl } \vec{F} = \nabla \times \vec{F} = \nabla \times \nabla \varphi = 0$.

So $\oint_C \vec{F} \cdot d\vec{r} = 0$ ∴ circulation is zero on all closed curves

· no work is done in moving an object on a closed path.

(Old Final Exam Sp 21)

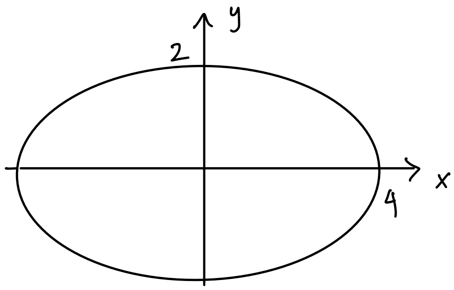
9. (10 Pts) Given the vector field $\vec{F} = \langle z-y, x+y, z\rangle$, use Stokes' Theorem to compute $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$ where S is that portion of the elliptic paraboloid $z = 4 - \frac{x^2}{4} - y^2$ that is above the xy-plane. Assume that $\vec{n}$ points upward.

One of the ways to "use Stokes' Thm" is to convert difficult surface integral into easier line integral:

Stokes' Theorem says $\iint_S (\underbrace{\nabla \times \vec{F}}_{\text{curl}}) \cdot \vec{n}\, dS = \oint_C \vec{F} \cdot d\vec{r}$ where C is the boundary of S.

C is the intersection of S and the xy-plane $(z=0)$:

$$0 = 4 - \frac{x^2}{4} - y^2 \Rightarrow 4 = \frac{x^2}{4} + y^2 \Rightarrow 1 = \frac{x^2}{16} + \frac{y^2}{4}$$



A parametrization of C is

$\vec{r}(t) = \langle 4\cos t, 2\sin t, 0\rangle$

for $0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -4\sin t, 2\cos t, 0\rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \underbrace{(z-y)}_{0-2\sin t}(-4\sin t) + \underbrace{(x+y)}_{4\cos t + 2\sin t}(2\cos t) + z\,(0)\ dt$$

$$= \int_0^{2\pi} \underbrace{8\sin^2 t + 8\cos^2 t}_{8} + 4\,\underbrace{\sin t\cos t}_{\frac{\sin(2t)}{2}}\ dt$$

(or use u-substitution)

$$= \int_0^{2\pi} 8 + 2\sin(2t)\, dt = 8t - \frac{2}{2}\cos(2t)\Big|_{t=0}^{t=2\pi}$$

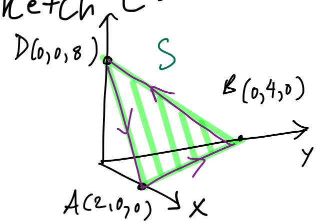
$$= 8(2\pi) - \underbrace{\cos(4\pi)}_{1} - \Big[0 - \underbrace{\cos(0)}_{1}\Big] = \boxed{16\pi}$$

Ex 2 Use Stokes' Theorem to compute the line integral $\oint_C \vec{F}\cdot d\vec{r}$ (circulation) of the vector field $\vec{F} = z\hat{i} - z\hat{j} + (x^2 - y^2)\hat{k}$, where C is the closed triangular path starting at point $A(2,0,0)$, to $B(0,4,0)$, to $D(0,0,8)$ and back to $A(2,0,0)$.

Group Quiz

Sol: $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot n \, dS$ (Stokes' Thm)

Sketch C:



Choose a surface S such that C is the boundary. Since C lives in a plane, we can choose S to be the plane containing A, B, D.

Review Sec 13.5 Lines & planes:

To find a normal vector to S, take $\vec{AB} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 4 & 0 \\ -2 & 0 & 8 \end{vmatrix}$

$= \hat{i}(32-0) - \hat{j}(-16-0) + \hat{k}(0--8) = \langle 32, 16, 8 \rangle$

Equation of plane S is: (normal vector) $\cdot \vec{AP} = 0$ — can choose B or D instead

$\langle 32, 16, 8\rangle \cdot \langle x-2, y-0, z-0\rangle = 0$

$32x - 64 + 16y + 8z = 0$

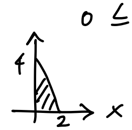
$4x - 8 + 2y + z = 0$

find $0 = 8 - 4x - 2y$ by setting $z = 0$

S: $z = 8 - 4x - 2y$ above the 2D region $R = \{(x,y) : 0 \le y \le -2x + 4,\ 0 \le x \le 2\}$.

Let $g(x,y) = 8 - 4x - 2y$

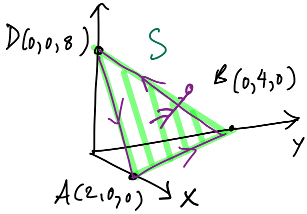
(Explicit description)



Compute normal vector of S: $\langle -z_x, -z_y, 1\rangle = \langle +4, +2, 1\rangle$

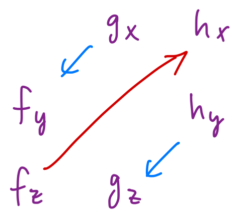
This vector is pointing "up", consistent w/ orientation of C (using right-hand rule)

so use this.



$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \langle -2y+1,\ 1-2x,\ 0-0\rangle = \langle -2y+1,\ 1-2x,\ 0\rangle$$

$$\vec{F} = \langle \underset{f}{z},\ \underset{g}{-z},\ \underset{h}{x^2+y^2}\rangle$$



$$\iint_S (\nabla \times \vec{F})\cdot\vec{n}\, dS = \iint_R \langle -2y+1, 1-2x, 0\rangle \cdot \langle 4, 2, 1\rangle\, dA$$

$$= \int_0^2 \int_0^{-2x+4} (-8y+4)+(2-4x)\, dy\, dx$$

inner:

$$\int_0^{-2x+4} -8y+6-4x\, dy = -8\frac{y^2}{2} + 6y - 4xy \Big|_{y=0}^{y=-2x+4} = -4y^2 + y(6-4x)\Big|_{y=0}^{y=-2x+4}$$

$$= -4(-2x+4)^2 + (-2x+4)(6-4x)$$

$$= -4(4x^2-16x+16) - 12x + 8x^2 + 24 - 16x$$

$$= -16x^2 + 64x - 64 - 28x + 8x^2 + 24$$

$$= -8x^2 - 36x - 40$$

skip

outer: $$\int_0^2 -8x^2+36x-40\, dx = -8\frac{x^3}{3} + 36\frac{x^2}{2} - 40x \Big|_{x=0}^{x=2} = \boxed{-\frac{88}{3}}$$

Negative circulation means net circulation is the opposite (clockwise) direction of C, looking from above.