17.6 Surface integrals

I. Parametrized surfaces

* A parametrization of a curve C in $\mathbb{R}^2$:

$$\vec{r}(t) = \langle x(t),\ y(t) \rangle \text{ for } a \le t \le b.$$

one parameter t

two dependent variables

* A parametrization of a surface S in $\mathbb{R}^3$:

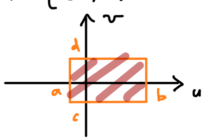
$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$

Two parameters u, v

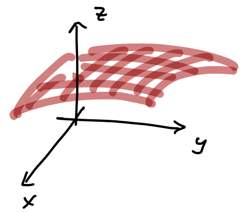
Three dependent variables

where the parameters u and v vary over a rectangle

$R = \{(u,v) : a \le u \le b,\ c \le v \le d\}$



each point (u,v) in R is sent to a point $\vec{r}(u,v)$ on S



(Note: parametric description of a surface is not unique, just like parametric description of a curve is not unique)

Examples of parametrized surfaces

1. Cylinder S $\{(x,y,z) : x = 5\cos\theta,\ y = 5\sin\theta,\ \underbrace{0 \le \theta \le \pi}_{}\,,\ 2 \le z \le 6\}$

(MML # 1)

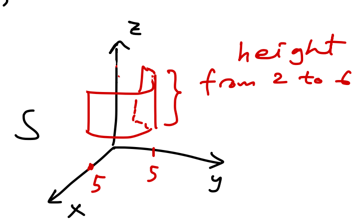
half a cylinder, 1st and 2nd quadrant only

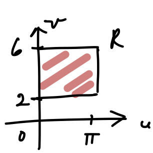
A possible parametrization of S is to let $u = \theta$, $v = z$

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$

$$= \langle 5\cos u,\ 5\sin u,\ v \rangle$$

for $0 \le u \le \pi$ and $2 \le v \le 6$





2. Plane S $\{ (x,y,z) : 3x - 2y + z = 2 \}$

Let $u = x$, $v = y$, and $z = -3x + 2y + 2$

$z = -3u + 2v + 2$

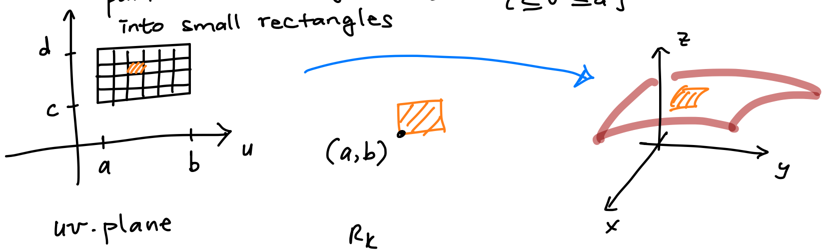
x and y can be any number

A possible parametrization of S is

$\vec{r}(u,v) = \langle \underbrace{u}_{x}, \underbrace{v}_{y}, \underbrace{-3u + 2v + 2}_{z} \rangle$ for $-\infty < u < \infty$ and $-\infty < v < \infty$.

II. Surface integral of a scalar-valued function $\iint_S f(x,y,z)\, dS$

Idea: partition rectangle $R = \{(u,v) : a \le u \le b,\ c \le v \le d\}$ into small rectangles



uv-plane

Each small rectangle R_k gets

sent by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

to a curved patch S_k on the surface S.

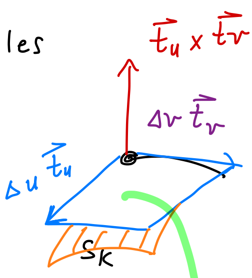
Let $P(a,b)$ be the lower left corner point of the small rectangle.

$\vec{r}(u,v)$ sends $P(a,b)$ to a point $P(x(a,b), y(a,b), z(a,b))$ on S_k.

Take the sum of $f(\ \ )$ over all lower left corners of the small rectangles R_k multiplied by area of S_k

$$\sum_{\substack{\text{over all} \\ \text{small rectangles}}} f(x(a_k,b_k),\ y(a_k,b_k),\ z(a_k,b_k))\ (\text{area of } S_k)$$

area of S_k



$\vec{t}_v = \dfrac{\partial \vec{r}}{\partial v} = \left\langle \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial v} \right\rangle$

tangent vector corresponding to change in v (keeping u constant)

$\vec{t}_u = \dfrac{\partial \vec{r}}{\partial u}$

$= \left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u} \right\rangle$

tangent vector corresponding to change in u (keeping v constant)

this parallelogram in the tangent plane has area $|\Delta u\, \vec{t}_u \times \Delta v\, \vec{t}_v|$

perә'le lә,gram

$= |\vec{t}_u \times \vec{t}_v|\, \Delta u\, \Delta v$

Def

- The surface integral of $f(x,y,z)$ on surface S, denoted $\iint_S f(x,y,z)\, dS$ is the limit of the sum as the sizes of the small rectangles go to 0.

- To compute surface integral, we first convert it to a double integral over the rectangle R:

(i) using a parametric description of surface S:

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_R f(x(u,v), y(u,v), z(u,v)) \left|\vec{t}_u \times \vec{t}_v\right| dA}$$

(ii) If S is explicitly defined by $z = g(x,y)$ for (x,y) in region R, then

$$\vec{t}_u = \vec{t}_x = \langle 1, 0, z_x \rangle$$
$$\vec{t}_v = \vec{t}_y = \langle 0, 1, z_y \rangle$$

$$\vec{t}_x \times \vec{t}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = \langle -z_x, -z_y, 1 \rangle$$

$$\left|\vec{t}_x \times \vec{t}_y\right| = \sqrt{z_x^2 + z_y^2 + 1}$$

So $$\boxed{\iint_S f\, dS = \iint_R f(x, y, g(x,y)) \sqrt{z_x^2 + z_y^2 + 1}\, dA}$$

Both are the usual 2D double integral from sec 16.2-16.3.

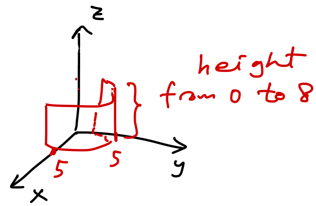
- If $f(x,y,z) = 1$, then $\iint_S f(x,y,z)\, dS =$ surface area of S

Ex 2(a) and MML # 3.

Find the area of the surface of the half cylinder

$\{(r,\theta,z): r=7,\ 0\le\theta\le\pi,\ 0\le z\le 8\}$

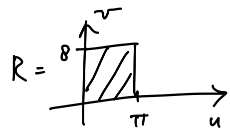
using a parametric description of S.



Sol: Parametrize S:

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$$
$$= \langle 7\cos u,\ 7\sin u,\ v\rangle$$

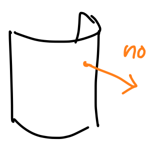
for $0\le u\le\pi$ and $0\le v\le 8$



$$\text{Surface Area} = \iint_S 1\, dS = \iint_R 1\, |\vec{t}_u\times\vec{t}_v|\, dA$$

$$\vec{t}_u\times\vec{t}_v = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ x_u & y_u & z_u\\ x_v & y_v & z_v\end{vmatrix} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -7\sin u & 7\cos u & 0\\ 0 & 0 & 1\end{vmatrix} = \hat{\imath}(7\sin u) - \hat{\jmath}(-7\cos u) + \hat{k}\,0$$
$$= \underbrace{\langle -7\sin u,\ 7\cos u,\ 0\rangle}$$

$$|\vec{t}_u\times\vec{t}_v| = \sqrt{7^2\sin^2 u + 7^2\cos^2}$$
$$= \sqrt{7^2(1)} = 7$$



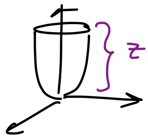
$$\text{surface area is } \iint_R 7\, dA = \int_0^8\int_0^\pi 7\, du\, dv = \boxed{56\pi}$$

MML #6 (Additional Ex)

call this surface S

Find the area of the surface $z = 5x^2 + 5y^2$ for $0 \le z \le 80$.

Sol: S is a paraboloid



z is from 0 to 80

Area of S is the surface integral $\iint\limits_S 1\, dS$

Projection of S on the xy-plane is

$$R = \{(x,y) : \underbrace{5x^2 + 5y^2 \le 80}_{x^2 + y^2 \le 16}\} = \{(r,\theta) : 0 \le r \le 4,\ 0 \le \theta \le 2\pi\}$$

Since S is explicitly defined by $z = 5x^2 + 5y^2$, $0 \le z \le 80$,

let $g(x,z) = 5x^2 + 5y^2$

Then $z_x = 10x$, $z_y = 10y$, so $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{100(x^2+y^2)+1}$

$$\iint\limits_S 1\, dS = \iint\limits_R 1 \sqrt{100\underbrace{(x^2+y^2)}_{r^2}+1}\, dA = \int_0^{2\pi}\int_0^4 \sqrt{100r^2+1}\; r\, dr\, d\theta$$

inner: $\displaystyle\int_0^4 \sqrt{100r^2+1}\; r\, dr = \frac{1}{200}\int_{u=1}^{u=1601} u^{\frac{1}{2}}\, du = \frac{1}{200}\cdot\frac{2}{3} u^{\frac{3}{2}} \Big|_{u=1}^{u=1601} = \frac{1}{300}\left[(1601)^{\frac{3}{2}} - 1\right]$

$u = 100r^2 + 1$

$du = 200r\, dr$

$\frac{1}{200} du = r\, dr$

outer: $\displaystyle\int_0^{2\pi} \frac{1}{300}\left[(1601)^{\frac{3}{2}} - 1\right] d\theta = \boxed{\frac{1}{300}\left[(1601)^{\frac{3}{2}} - 1\right] 2\pi}$

8. (10 Pts) If the density of ants on the surface of the paraboloid $z = x^2 + y^2$ $(0 \le z \le 4 \text{ m})$ is given by $f(x, y, z) = 32\,(x^2 + y^2)\left(\frac{ants}{m^2}\right)$, determine the total number of ants on the surface. *Note:* Focus on expressing the answer as a simplified integral in polar coordinates. Complete the integral if you can, but correct final integration will only be scored 1 point.

Group Quiz Fri

Ref: See Sec 17.6 Example 6, MML # 6

Mass on a surface S is the surface integral $\iint_S \begin{pmatrix}\text{density}\\ \text{function}\end{pmatrix} dS = \iint_S f(x,y,z)\, dS$

Since the surface is explicitly defined by $z = \underbrace{x^2 + y^2}$, $0 \le z \le 4$,

let $g(x,z) = x^2 + y^2$

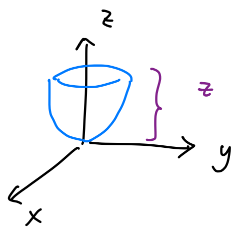
we will use the double integral

THEOREM 17.14 Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces
Let f be a continuous function on a smooth surface S given by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, g(x, y))\sqrt{z_x^2 + z_y^2 + 1}\, dA.$$

If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

Sketch of S:



z is from 0 to 4

Set $z = 4$:
$x^2 + y^2 = 4$

The projection of S on the xy-plane is $R = \{(x,y) : x^2 + y^2 \le 4\} = \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

$z_x = 2x$, $z_y = 2y$, so $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$

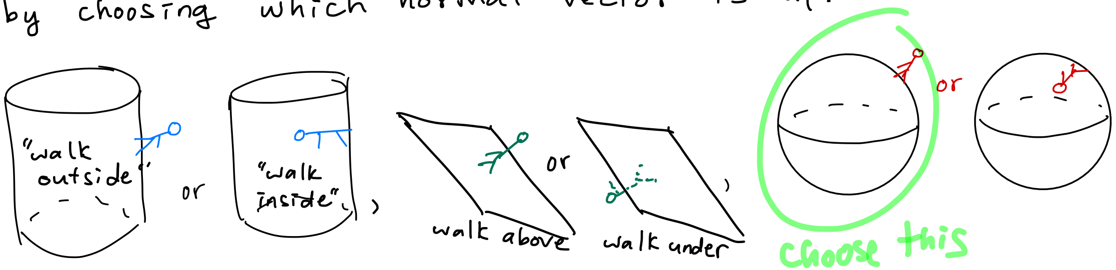
$$\iint_S f(x,y,z)\, dS = \iint_R 32\,\underbrace{(x^2+y^2)}_{r^2}\sqrt{\underbrace{4x^2+4y^2}_{4r^2}+1}\; dA$$

write the iterated integral in polar because R is a disk (See Sec 16.3 "Double integrals in polar coordinates")

$$= 32\int_0^{2\pi}\int_0^{2} r^2\sqrt{4r^2+1}\;\overset{\text{extra}}{r}\; dr\, d\theta$$

III Surface integral of a vector field.

Given a surface S, we turn S into an <u>oriented surface</u> by choosing which normal vector is "up".



When the surface is closed (e.g. a sphere), we choose the orientation so that the normal vectors point <u>outward</u>.

<u>Def</u> The <u>flux integral</u> of a vector field $\vec{F} = \langle f, g, h \rangle$ across an oriented surface S, denoted $\iint_S \vec{F} \cdot \vec{n} \, dS$ is the net flow across the surface.

Ex. water that passes through barrier of a cell during 1 minute

(flow into the "up" side of S gives positive contribution, flow into the opposite side of S gives negative contribution)

To compute, first convert into a double integral:

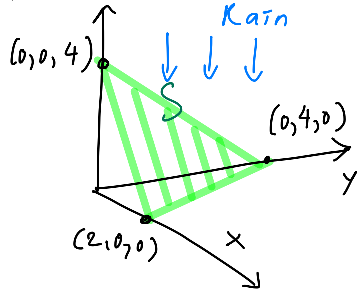
using parametric description of surface S

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_R \vec{F} \cdot (\vec{t}_u \times \vec{t}_v) \, dA$$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_R \langle f, g, h \rangle \cdot \langle -z_x, -z_y, 1 \rangle \, dA$$

Ex 7 (Rain on a roof) (Additional Ex)

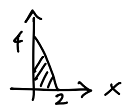
A constant rain can be modeled by the vertical vector field $\vec{F} = \langle 0, 0, -1 \rangle$. A roof can be modeled by S where S is the plane $z = 4 - 2x - y$ in the positive octant. Find the flux in the downward direction across S.



(Application: if $\vec{F}$ is rate of rainfall per min, this flux integral gives the mass of rain that falls on the roof in one minute)

Sol: Surface S is given explicitly $z = \overbrace{4 - 2x - y}^{\text{let } g(x,y) = 4 - 2x - y}$

for (x, y) in the region R



$R = \{ (x,y) : 0 \le y \le -2x + 4, \; 0 \le x \le 2 \}$.

$$\vec{t}_x \times \vec{t}_y = \langle -z_x, -z_y, 1 \rangle = \langle 2, 1, 1 \rangle.$$

(z-component is positive).

This vector points upward

Because we want downward flux, take the same vector but in opposite direction $\langle -2, -1, -1 \rangle$.

$$\text{Flux is } \iint_S \vec{F} \cdot \vec{n} \, dS = \iint_R \underbrace{\langle 0, 0, -1 \rangle \cdot \langle -2, -1, -1 \rangle}_{0 + 0 + (-1)(-1)} dA = \iint_R 1 \, dA = (\text{area of } R) = \frac{2(4)}{2}$$

Observe: mass of rain that falls on the roof

= mass of rain that would fall on the floor beneath the roof if the roof were not there.

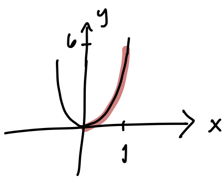
— the end —

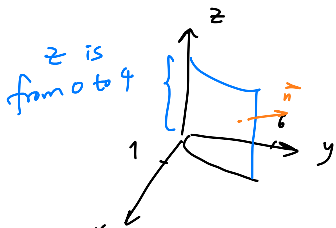
MML # 7

Find the flux $\iint_S \vec{F} \cdot \vec{n}\, dS$ where $\vec{F} = \langle -y, x, 1 \rangle$ and

S is cylinder $y = 6x^2$ for $0 \le x \le 1$, $0 \le z \le 4$,

normal vectors point in the general direction of the positive y-axis





parametrization of S: Let $x = u$, $z = v$

$\vec{r}(u,v) = \langle u, 6u^2, v \rangle$ for $0 \le u \le 1$, $0 \le v \le 4$

$\vec{t}_u = \langle 1, 12u, 0 \rangle$

$\vec{t}_v = \langle 0, 0, 1 \rangle$

$$\vec{t}_u \times \vec{t}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1, & 12u, & 0 \\ 0, & 0, & 1 \end{vmatrix} = \hat{\imath}(12u) - \hat{\jmath}(1) + \hat{k}(0)$$

$$= \langle 12u, -1, 0 \rangle$$

This is the opposite of what we want: $\langle -12u, 1, 0 \rangle$

flux is $\iint_S \langle -y, x, 1 \rangle \cdot \vec{n}\, dS = \int_0^4 \int_0^1 \langle -6u^2, u, 1 \rangle \cdot \langle -12u, 1, 0 \rangle \, du\, dv$

$$= \int_0^4 \int_0^1 72u^3 + u \quad du\, dv$$

inner: $\int_0^1 72u^3 + u \quad du = 72\frac{u^4}{4} + \frac{u^2}{2} \Big|_{u=0}^{u=1} = 18(1) + \frac{1}{2} = \frac{37}{2}$

outer: $\int_0^4 \frac{37}{2}\, dv = \frac{37}{2}(4) = \boxed{74}$

— end —