17.5 Divergence and curl

Goal of Sec 17.5–17.8: Lift Green's thm (circulation form & flux form) from $\mathbb{R}^2$ into $\mathbb{R}^3$.

circulation form: using ↺ curl

flux form: using divergence

Now: Extend def of divergence (Part I) and curl (Part II) to $\mathbb{R}^3$.

Def

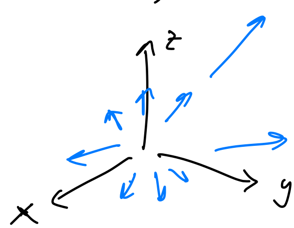
- Let $\vec{F} = \langle f, g, h \rangle$ be a vector field which is differentiable on a region of $\mathbb{R}^3$.
- Recall the del operator $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ "nabla"

Define $\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

"dot product of ∇ and $\vec{F}$"

- The divergence of $\vec{F}$ is $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$.
- Like in $\mathbb{R}^2$, if $\nabla \cdot \vec{F} = 0$ for all points in $\mathbb{R}^3$, we say $\vec{F}$ is source free.

Ex 1: (a) $\vec{F} = \langle x, y, z \rangle$ "radial vector field"



$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

for all points (a, b, c) in $\mathbb{R}^3$.

So flow expands outward at all points

Note: $\operatorname{div} \vec{F}$ at a point $P(a,b,c)$ measures expansion / contraction of $\vec{F}$ at that point.

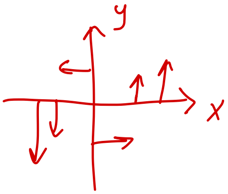
Ex 1(c)

$\vec{F} = \langle \underbrace{-y, x}_{}, z \rangle$ "spiral flow"

the 2D rotational field $\langle -y, x \rangle$

vertical flow in the z-direction

if you view the field from above



divergence
$f_x + g_y = 0$
at all points

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z} z$$

dot product

$$= \underbrace{0 + 0}_{} + 1$$

Rotational part of $\vec{F}$ doesn't contribute to divergence

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Part II The curl

In 2D: The curl of $\vec{F} = \langle f, g \rangle$ is $g_x - f_y$ (prev sec 17.4)

In 3D: The curl of $\vec{F} = \langle f, g, h \rangle$ is curl $\vec{F}$ =

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y} h - \frac{\partial}{\partial z} g\right) - \hat{\jmath}\left(\frac{\partial}{\partial x} h - \frac{\partial}{\partial z} f\right) + \hat{k}\left(\frac{\partial}{\partial x} g - \frac{\partial}{\partial y} f\right)$$

$$= \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$$

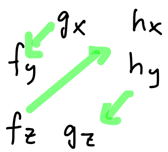
this 2D curl gives the rotation in the xy-plane

Def If the curl of $\vec{F}$ is $\langle 0, 0, 0 \rangle$, the vector field is called irrotational (as in $\mathbb{R}^2$ case)

MML #6

Compute the curl of $\vec{F} = \langle \underbrace{4x^2 - 5y^2}_{f}, \underbrace{2xy}_{g}, \underbrace{z}_{h} \rangle$

Sol:



$$\nabla \times \vec{F} = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$$

$$= \langle 0 - 0, 0 - 0, 2y - (-10y) \rangle$$

$$= \boxed{12y\,\hat{k}}$$

Thm (The curl of a conservative vector field is the zero vector)

If $\vec{F}$ is conservative ($\vec{F} = \nabla\varphi$ for some φ),

then curl $\vec{F} = \nabla \times \vec{F} = \langle 0,0,0 \rangle$

Why? $h_y = g_z$, $f_z = h_x$, $g_x = f_y$ are the conditions needed in the Test for Conservative Vector Fields (Sec 17.3)

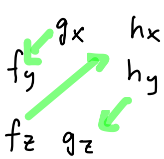
Thm (Divergence of the curl is zero)

Let $\vec{F} = \langle f, g, h \rangle$

$$\nabla \cdot (\nabla \times F) = 0$$

Why?

$$\nabla \cdot \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle = \frac{\partial}{\partial x}(h_y - g_z) + \frac{\partial}{\partial y}(f_z - h_x) + \frac{\partial}{\partial z}(g_x - f_y)$$



$$= h_{yx} - g_{zx} + f_{zy} - h_{xy} + g_{xz} - f_{yz}$$

$= 0$ because mixed partial derivatives are equal (Sec 15.3) if they are continuous

$F_{ab} = F_{ba}$