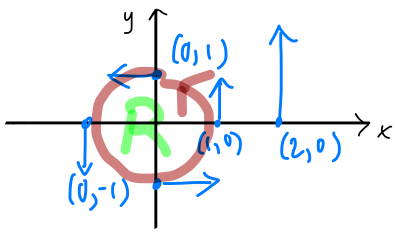
17.4 Green's Thm

Ex "rotation" vector field $\vec{F} = \langle -y, x \rangle$ wind pattern on the xy-plane ($f = -y$, $g = x$)



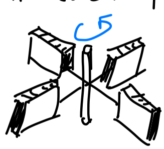
$C: \vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le 2\pi$

you're walking along C

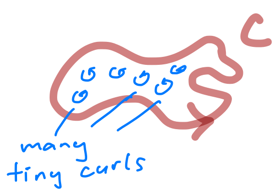
Circulation $\oint_C \vec{F} \cdot d\vec{r} = \oint_C f\,dx + g\,dy$ is the net amount of tailwind helping you move (positive contribution) & headwind (negative contribution)

We computed $\oint_C \vec{F} \cdot d\vec{r} = 2\pi$ in Sec 17.2

A nonzero circulation on a closed curve C means that, inside the curve, $\vec{F}$ must have many "tiny circulation" that produces the circulation: At each point (a,b) in R, imagine a paddle wheel



with certain speed and direction.



Assume C is simple, closed, piecewise-smooth curve, oriented counter clockwise. Assume R is the inside of C, and R is connected and simply connected.

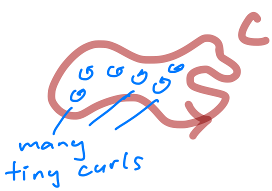
Assume $\vec{F} = \langle f, g \rangle$, and f & g have continuous partial derivatives in R.

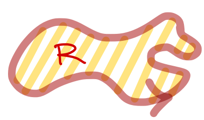
Then we have ...

Green's Thm – Circulation Form (Part I)

$$\underbrace{\oint_C \vec{F}\cdot d\vec{r} = \oint_C f\,dx + g\,dy}_{\text{circulation}} = \iint_R \boxed{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}\, dA$$

called the <u>two-dimensional curl</u> of $\vec{F}$





<u>Def</u> $\vec{F}$ is called <u>irrotational</u> if the curl $g_x - f_y$ is 0 for all pt (a,b) in R.

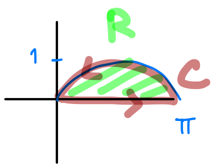
<u>Note</u> If $\vec{F}$ is conservative, $g_x = f_y$ so $\oint_C \vec{F}\cdot d\vec{r} = 0$.

(when $\vec{F}$ is not conservative, we can use Green's thm to simplify computation)

Ex:

$\vec{F} = \langle \underbrace{-3y}_{f}, \underbrace{4x}_{g} \rangle$ vector field

R the region bounded by $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$.



- What is the two-dimensional curl of $\vec{F}$ in R?

Sol: $g_x - f_y = 4 - (-3) = 7$ (Note: $f_y \neq g_x$ means $\vec{F}$ is not conservative)

- Let C be the boundary of R, oriented counterclockwise. Compute the circulation of $\vec{F}$ across C.

Sol: We can compute the circulation $\oint_C \vec{F} \cdot d\vec{r} = \oint f\,dx + g\,dy$ using Sec 17.2 method.

Or we can use Green's Theorem: → will use today

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\; dA = \iint_R 7\; dA$$

$$= \int_0^{\pi} \int_0^{\sin x} 7\; dy\, dx = 14$$

inner $\displaystyle\int_0^{\sin x} 7\, dy = 7y \Big|_{y=0}^{y=\sin x} = 7 \sin x$

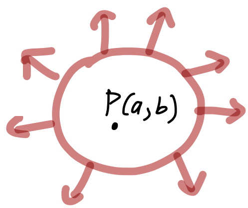
outer $\displaystyle\int_0^{\pi} 7 \sin x\, dx = -7 (\cos x) \Big|_{x=0}^{x=\pi} = -7(-1-1) = \boxed{14}$

Green's Thm - Flux Form (Part II)

$$\underbrace{\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C f\,dy - g\,dx}_{\text{outward flux}} = \iint_R \boxed{\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)}\, dA$$

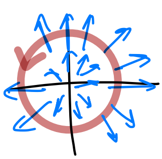
called the two-dimensional divergence of $\vec{F}$

If $f_x + g_y > 0$ at a point $P(a,b)$ in R, we get a net outward flux across a small circle enclosing $P(a,b)$



Def $\vec{F}$ is called source free on R if the divergence $f_x + g_y$ is 0 for each point $P(a,b)$ in R.

Ex 3 Use Green's Thm to compute the outward flux of the 2-D "radial" vector field $\vec{F} = \langle \overset{f}{x}, \overset{g}{y} \rangle$ through the unit circle $C = \{(x,y) : x^2 + y^2 = 1\}$. R is the disk inside C



Note: In Sec 17.2 we computed the outward flux

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C f\,dy - g\,dx \text{ to be } 2\pi$$

Sol: The divergence of $\vec{F}$ is $f_x + g_y = 1 + 1 = 2$.

By Green's Thm, (flux form) $\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \underbrace{\text{divergence}}_{f_x+g_y}\, dA = \iint_R 2\,dA = 2$ area of R

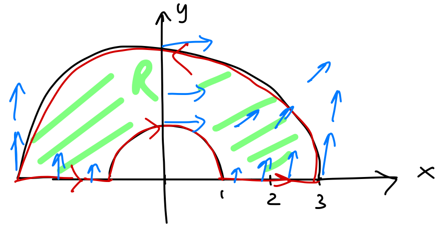
$= 2(\pi 1^2)$

$= 2\pi$ ✓

Ex 5:

$$\vec{F} = \langle \underset{f}{y^2}, \underset{g}{x^2} \rangle$$

$R = \{ (x,y) : 1 \le x^2 + y^2 \le 9, \ y \ge 0 \}$ half-annulus



Let C be the boundary of R, oriented counterclockwise (so that the inside of C is always to your left as you follow C) Find the circulation of $\vec{F}$ on C.

Sol We can do this directly (Sec 17.2) but we will need to compute four integrals.

Let's use Green's Thm - circulation form:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \underbrace{y^2}_{f} dx + \underbrace{x^2}_{g} dy = \iint_R \overbrace{\underbrace{2x}_{g_x} - \underbrace{2y}_{f_y}}^{\text{curl}} \, dA$$

$$= 2 \iint_R x - y \, dA$$

$x = r\cos\theta$
$y = r\sin\theta$
Convert to polar coordinates

$$= 2 \int_0^{\pi} \int_1^3 (r\cos\theta - r\sin\theta) \underset{\text{extra}}{r} \, dr \, d\theta$$

$$= 2 \int_0^{\pi} (\cos\theta - \sin\theta) \left(\int_1^3 r^2 \, dr \right) d\theta$$

inner: $\int_1^3 r^2 \, dr = \frac{r^3}{3} \Big|_1^3 = \frac{1}{3}(27-1) = \frac{26}{3}$

outer: $2 \int_0^{\pi} \frac{26}{3} (\cos\theta - \sin\theta) d\theta = \frac{52}{3} \left(\sin\theta + \cos\theta \right]_{\theta=0}^{\theta=\pi} = \frac{52}{3}(-1-1) = \boxed{-\frac{104}{3}}$ //