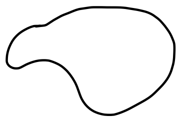
17.3 Conservative vector fields

Let C be a curve w/ parametrization $\vec{r}(t)$ for $a \le t \le b$.

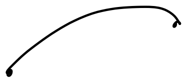
Def C is called ...

- **simple** if C never intersects itself
- **closed** if the initial and terminal points $\vec{r}(a)$ and $\vec{r}(b)$ are the same

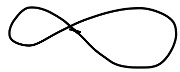
Ex



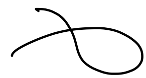
closed
simple



not closed
simple



closed
not simple



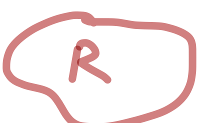
not closed
not simple

Let R be an open region in $\mathbb{R}^2$.

Def • R is called **connected** if it's possible to connect any two points of R by a curve lying in R.

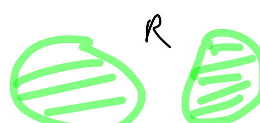
• R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R.

Ex



connected,
simply connected

connected,
not simply connected



not connected,
simply connected

In this section, we assume our vector field $\vec{F}$ is defined on a region that is both connected and simply connected.

Ex $\mathbb{R}^2$ and $\mathbb{R}^3$ are connected & simply connected.

Def A vector field $\vec{F}$ is called conservative on a region R if $\vec{F} = \nabla \varphi$ for some scalar function on that region.

Thm (Test for conservative vector fields)

Suppose $\vec{F} = \langle f, g, h \rangle$ is a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$.

$\vec{F}$ is a conservative vector field on D iff

$$f_y = g_x \qquad f_z = h_x \qquad g_z = h_y$$

$$\begin{matrix} f_x & g_x & h_x \\ f_y & g_y & h_y \\ f_z & g_z & h_z \end{matrix}$$

For $\vec{F} = \langle f, g \rangle$ in $\mathbb{R}^2$, we just need to check whether $f_y = g_x$

Ex 1 (a) and Ex 2(a)

(Additional Example)

$$\vec{F} = \langle \underbrace{e^x \cos y}_{f(x,y)}, \underbrace{-e^x \sin y}_{g(x,y)} \rangle$$

$$f_y = -e^x \sin y \quad , \quad g_x = -e^x \sin y$$

Since $f_y = g_x$, the test for conservative vector fields tells us $\vec{F}$ is conservative.

So $\vec{F} = \nabla \varphi$ for some scalar function $\varphi(x,y)$.

How do we find φ? Let $\varphi_x = f = e^x \cos y$

$\varphi_y = g = -e^x \sin y$

1. $\varphi_x = f = e^x \cos y.$

Take antiderivative wrt x:

$$\int \varphi_x \, dx = \int e^x \cos y \, dx = e^x \cos y + \underbrace{c(y)}_{\text{an arbitrary function of } y}$$

So $\varphi(x,y) = e^x \cos y + c(y)$

2. Differentiate the expression from step 1 wrt y:

$$\varphi_y = -e^x \sin y + c'(y)$$

Set this equal to g:

$$-e^x \sin y + c'(y) = -e^x \sin y \Rightarrow c'(y) = 0$$

$c(y) =$ some number

I choose $c = 0$.

So $\boxed{\varphi = e^x \cos y}$

Confidence check: $\nabla \varphi(x,y) \stackrel{?}{=} \vec{F}$

Ex 1(b) and 2(b)

$$\vec{F} = \langle \underbrace{2xy - z^2}_{f}, \underbrace{x^2 + 2z}_{g}, \underbrace{2y - 2xz}_{h} \rangle$$

$g_x = 2x$ $\quad h_x = -2z$

$f_y = 2x$ $\quad h_y = 2$

$f_z = -2z$ $\quad g_z = 2$

$\vec{F}$ satisfies the three conditions for the Test for Conservative Vector Fields, so $\vec{F}$ is conservative.

So $\vec{F} = \nabla\varphi$ for some scalar function $\varphi(x,y,z)$

How do we find φ?

⓪ Want $\varphi_x = f = 2xy - z^2$, $\varphi_y = g = x^2 + 2z$, $\varphi_z = h = 2y - 2xz$.

① Take antiderivative of $\varphi_x = f$ wrt x:

$$\varphi = \int 2xy - z^2 \, dx = 2\frac{x^2}{2}y - z^2 x + \underline{c(y,z)}$$

the arbitrary "constant" is a function of y and z

Note: At this point, $\varphi(x,y,z)$ looks like $x^2y - z^2x + c(y,z)$.

② Differentiate current φ wrt y: $\quad \varphi_y = x^2 + \frac{\partial}{\partial y}c(y,z)$

Set it equal to g: $\quad x^2 + 2z = x^2 + \frac{\partial}{\partial y}c(y,z)$

find $\frac{\partial}{\partial y}c$: $\quad \frac{\partial}{\partial y}c(y,z) = 2z$

3 To find $c(y,z)$, take antiderivative of $\frac{\partial}{\partial y} c(y,z)$ wrt y:

$$c(y,z) = \int 2z \, dy = 2zy + \underbrace{d(z)}_{\text{the "constant" is a function of } z}$$

Note: At this point, $\varphi(x,y,z)$ looks like $x^2 y - z^2 x + \overbrace{2zy + d(z)}^{c(y,z)}$

4 Differentiate current φ wrt z : $\varphi_z = -2zx + 2y + d'(z)$

Set it equal to h: $2y - 2xz = -2zx + 2y + d'(z)$

find $d'(z)$: $d'(z) = 0$

So $d(z)$ can be any number

So let $d(z) = 0$ (for simplicity)

So a potential function that works is $\boxed{\varphi = x^2 y - z^2 x + 2zy}$

Check: $\nabla\varphi = \langle 2xy - z^2,\ x^2 + 2z,\ -2zx + 2y \rangle \stackrel{\checkmark}{=} \vec{F}$

Fundamental Thm (FT)

__Recall__ $\int_a^b f'(x)\,dx = f(b) - f(a)$ (FT of Calc II)

FT for line integrals

If a vector field $\vec{F}$ is conservative on R (meaning $\vec{F} = \nabla \varphi$ for some φ), then

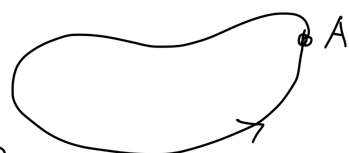
two ways of writing the line integral

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A)$$

for all points A, B in R and all piecewise-smooth oriented curves C in R from A to B.

__Observe__ Suppose $\vec{F}$ is conservative on R.

If C is a closed curve,



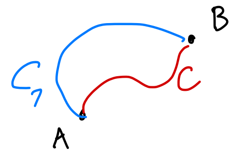
$$\oint_C \vec{F} \cdot d\vec{r} = \varphi(A) - \varphi(A) = 0$$

put this small circle to indicate C is a closed curve

Def A line integral $\int_C \vec{F}\cdot d\vec{r}$ is called <u>independent of path</u>

if $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r}$ for all piecewise-smooth curves C_1

w/ the same initial and terminal points.



Thm

If all line integrals of $\vec{F}$ $\left(\int_C \vec{F}\cdot d\vec{r}\right)$ are independent of path

then $\vec{F}$ is conservative (meaning $\vec{F} = \nabla\varphi$ for some potential function φ).

Summary of facts Suppose $\vec{F}$ is a vector field on an open region in $\mathbb{R}^2$ or $\mathbb{R}^3$

The following three properties are equivalent
(if you check that one property is true then we get the other two for free)

1) $\vec{F} = \nabla\varphi$ for some potential function φ
($\vec{F}$ is conservative)

2) $\int_C \vec{F}\cdot d\vec{r} = \varphi(B) - \varphi(A)$ for all points A, B in R and all piecewise-smooth oriented curves C in R from A to B
(all line integrals are path independent)

3) $\oint_C \vec{F}\cdot d\vec{r} = 0$ on all simple piecewise-smooth closed oriented curves C in R.

Ex 4: Given a force field $\vec{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$

Find the work required to move an object from the point $\underbrace{(-3,-2,-1)}_{A}$ to $\underbrace{(1,2,3)}_{B}$ following an oriented curve C.

Sol: In Ex 1&2(b), we checked that $\vec{F}$ is conservative and $\vec{F} = \nabla\varphi$ for $\varphi = x^2y - xz^2 + 2yz$.

By FT for line integrals,

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\varphi \cdot d\vec{r} = \varphi(B) - \varphi(A) = \varphi(1,2,3) - \varphi(-3,-2,-1)$$

$$= 2 - 3^2 + 2(2)(3) - \left[9(-2) - (-3)(1) + 2(-2)(-1)\right] = 16$$

By def, work is $\int_C \vec{F}\cdot d\vec{r} = \boxed{16}$ //