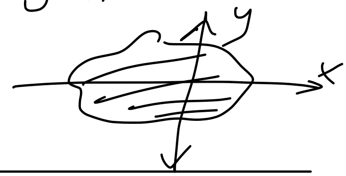
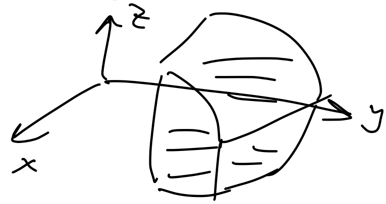
17.2 Line integrals

topic	notation	integrate over ...
Single-var Calc	$\int_a^b f(x)\,dx$	interval $[a,b]$ in real line $\mathbb{R}$
double integral	$\iint_R f(x,y)\,dA$	2D region R in $\mathbb{R}^2$ 
triple integral	$\iiint_D f(x,y,z)\,dV$	3D region D in $\mathbb{R}^3$ 

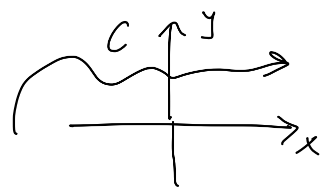
Application:
average, vol, area, mass

(New) Scalar Line integral (should really be called "curve integral" or "path integral")

$$\int_C f\,ds$$

a curve C in $\mathbb{R}^2$

$\vec{r}(t) = \langle x(t), y(t)\rangle$



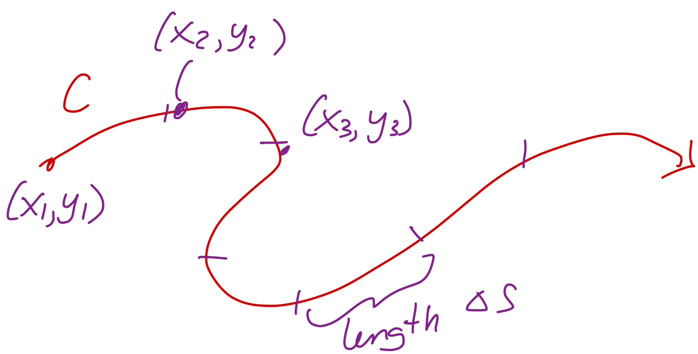
or curve C in $\mathbb{R}^3$

I. (Scalar) line integral

What is $\int_C f\, ds$?

$f(x,y)$ is a function w/ input points in $\mathbb{R}^2$ and outputs are numbers in $\mathbb{R}$

Cut C into n parts w/ equal length Δs, choose a representative point in each part (to make it simple, choose the earliest point



Take sum $\Delta s \left[f(x_1, y_1) + f(x_2, y_2) + \cdots \right]$

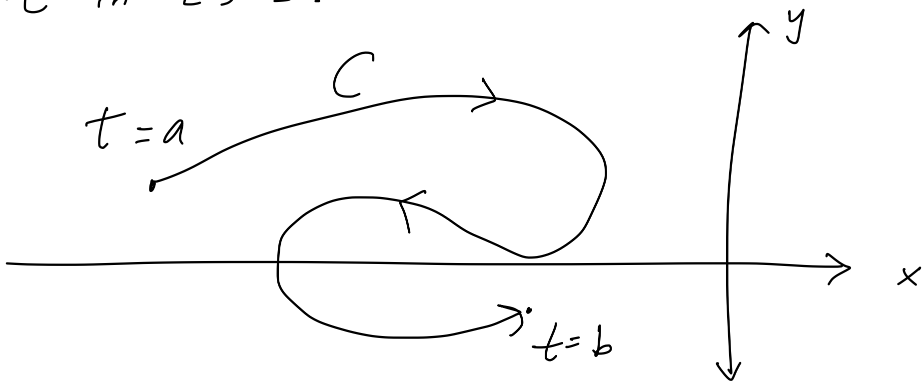
The limit of this sum as $n \to \infty$.

If the limit exists, it's denoted $\int_C f\, ds$.

We say f is <u>integrable</u> on C.

Recall Sec 14.4

Let a smooth curve C in $\mathbb{R}^2$ be represented by the parametric function $\vec{r}(t) = \langle f(t), g(t) \rangle$, for t in $[a, b]$.



The length of C over the interval $[a,b]$ is $\int_a^b |\vec{r}\,'(t)|\, dt$

Procedure for evaluating line integral $\int_C f\, ds$

① Choose a parametric description of C

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \le t \le b$.

② Compute $|\vec{r}\,'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

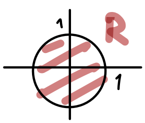
③
$$\boxed{\int_C f\, ds = \underbrace{\int_a^b f(x(t), y(t), z(t))\, \overbrace{|\vec{r}\,'(t)|\, dt}^{\text{Think } ds}}_{\text{single-variable integral}}}$$

Ex 1: The temperature of the circular plate

$R = \{(x,y) : x^2 + y^2 \le 1\}$ is $f(x,y) = 100\,(x^2 + 2y^2)$

Find the average temp along the perimeter of the plate.

Sol: R



Boundary of R is circle $C = \{(x,y) : x^2 + y^2 = 1\}$

Average value of f over C is $\dfrac{1}{\text{length of } C} \displaystyle\int_C f\, ds$

① Choose a parametric description of C:

$\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $\underset{a}{0} \le t \le \underset{b}{2\pi}$

② $\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = \sqrt{1} = 1$

3 $\int_C f\, ds = \int_0^{2\pi} \underbrace{100\left(x(t)^2 + 2y(t)^2\right)}_{f(t)} \underbrace{\left|\vec{r}\,'(t)\right|}_{1} \, dt$

$$= 100 \int_0^{2\pi} \cos^2(t) + 2\sin^2(t) \quad dt$$

$$= 100 \int_0^{2\pi} \underbrace{\left(\cos^2 t + \sin^2 t\right)}_{1} + \underbrace{\sin^2 t}_{\frac{1}{2} - \frac{\cos(2t)}{2}} \quad dt$$

$$= 100 \int_0^{2\pi} 1 + \frac{1}{2} - \frac{\cos(2t)}{2} \quad dt$$

$$= 100 \left[\left(\frac{3}{2}\right)t - \frac{1}{2}\,\frac{\sin(2t)}{2}\right]\Bigg|_{t=0}^{t=2\pi}$$

$$= 100\ \frac{3}{2}(2\pi) = 300\pi$$

Length of the circle is 2π (radius) $= 2\pi$

Average temp is $\frac{1}{2\pi}(300\pi) = \boxed{150}$

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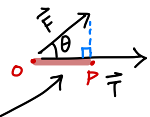
II. Line integrals of vector fields

Sec 13.3:



$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{|\vec{OP}|}{|\vec F|} \Rightarrow |\vec{OP}| = |\vec F| \cos\theta$

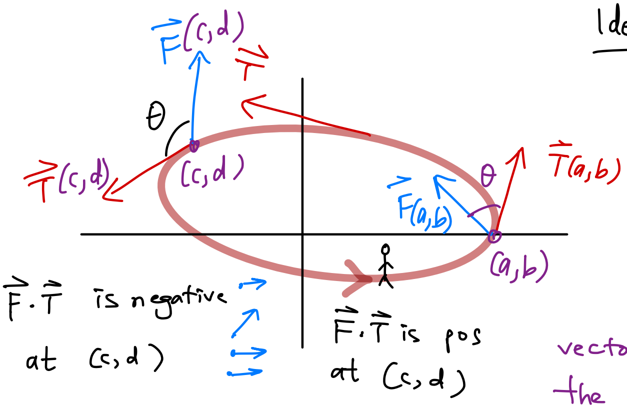
The (Scalar) component of $\vec F$ in the direction of $\vec T$ is $|\vec{OP}|$

If $\vec T$ is a unit vector, $|\vec T| = 1$, so $|\vec{OP}| = |\vec F|\,|\vec T| \cos\theta$
$= \vec F \cdot \vec T$

Now, let $\vec F(x,y)$ be a vector field,

let C be an oriented curve w/ parametrization $\vec r(t)$.

What is $\int_C \vec F \cdot \vec T \, ds$?



__Idea:__ Walk along C following the positive orientation. At each point (a,b) of C, compute $\underbrace{\vec F(a,b)}_{\text{vector given by the vector field}} \cdot \underbrace{\vec T(a,b)}_{\text{unit tangent vector of } \vec r(t) \text{ at } (a,b)}$

$\int_C \vec F \cdot \vec T \, ds$ is (the limit of) adding up the components of $\vec F$ in the direction of C at each point (a,b) of C.

This number doesn't depend on the choice of parametrization $\vec r(t)$ / how fast we are walking along C.

How to compute the line integral of $\vec{F}$ over C (symbol: $\int_C \vec{F}\cdot\vec{T}\, ds$):

Consider a vector field $\vec{F}(x,y) = \langle f(x,y), g(x,y)\rangle$

Consider a curve C with a parametrization $\vec{r}(t) = \langle x(t), y(t)\rangle$ for $a \le t \le b$.

$$\int_C \vec{F}\cdot\vec{T}\, ds = \int_a^b \vec{F}\cdot\vec{r}\,'(t)\, dt$$

sub $x(t)$ and $y(t)$ using formula from $\vec{r}(t)$

$$= \int_a^b \langle f(t), g(t)\rangle \cdot \langle x'(t), y'(t)\rangle\, dt$$

$$= \int_a^b \left[f(t)\,x'(t) + g(t)\,y'(t)\right] dt$$

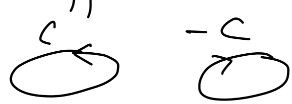
Note: Other notations for line integral of a vector field

$$\int_C \vec{F}\cdot\vec{T}\, ds = \int_C f\, dx + g\, dy \qquad \left(\text{Let } dx = x'(t)\, dt,\ dy = y'(t)\, dt\right)$$

$$= \int_C \vec{F}\cdot d\vec{r} \qquad \left(\text{let } d\vec{r} = \langle dx, dy\rangle\right)$$

Thm: $\int_{-C} \vec{F}\cdot\vec{T}\, ds = -\int_C \vec{F}\cdot\vec{T}\, ds$

where $-C$ means the same curve w/ opposite orientation



Application:

Amount of work needed to put a satellite into orbit (integrate gravitational force along path of satellite).

Applications

Def (Work done in a force field)

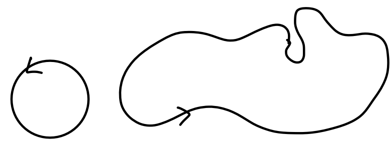
Let the vector field $\vec{F}$ be a force field (ex: a gravitational field),

C an oriented curve w/ parametrization $\vec{r}(t)$ for $a \le t \le b$.

Then the work done in moving an object along C is

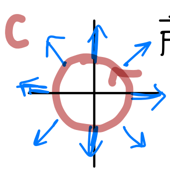
$$W = \underbrace{\int_C \vec{F}\cdot\vec{T}\,ds}_{\text{notation}} = \underbrace{\int_a^b \vec{F}\cdot\vec{r}\,'(t)\,dt}_{\text{How to compute}}$$

Def (Circulation)

If C is a closed oriented curve (like )

then the circulation of a vector field $\vec{F}$ on C is also $\int_C \vec{F}\cdot\vec{T}\,ds$.

Idea: As you travel along C, how much of the vector field $\vec{F}$ points in the direction of C. If $\vec{F}$ is wind pattern (that doesn't change with time), the circulation of $\vec{F}$ along C is the net amount of head wind (negative contribution) and tail wind (positive contribution).

Ex 6 (a) 

$\vec{F}(x,y) = \langle x, y\rangle$

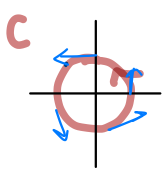
"radial" vector field"

Here $\int_C \vec{F}\cdot\vec{T}\,ds$ should be 0

$\vec{F}$ is always perpendicular to $\vec{T}$

unit tangent vector of $\vec{r}$

(b) $\vec{F}(x,y) = \langle -y, x\rangle$



"rotation vector field"

Here $\int_C \vec{F}\cdot\vec{T}\,ds$ should be positive because $\vec{F}$ is always pointing in the same direction of $\vec{T}$.

Q1: What is the circulation of the vector field $\vec{F} = \langle -y, x \rangle$ on C, the unit circle with counterclockwise orientation?

Q2: Given the force field $\vec{F} = \langle -y, x \rangle$, what is the work required to move an object from $(0,0)$ back to itself following C?

Answer: Q1 & Q2 have the same answer: $\int_C \vec{F} \cdot \vec{T} \, ds$

To compute $\int_C \vec{F} \cdot \vec{T} \, ds$, choose a parametrization of C:

$$\vec{r}(t) = \langle \overset{x}{\cos t}, \overset{y}{\sin t} \rangle \quad \text{for } 0 \leq t \leq 2\pi.$$

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_0^{2\pi} \vec{F} \cdot \vec{r}\,'(t) \, dt$$

$$= \int_0^{2\pi} \underbrace{\langle -\sin t, \cos t \rangle}_{\vec{F} = \langle -y, x \rangle} \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{2\pi} (\sin t)^2 + (\cos t)^2 \, dt$$

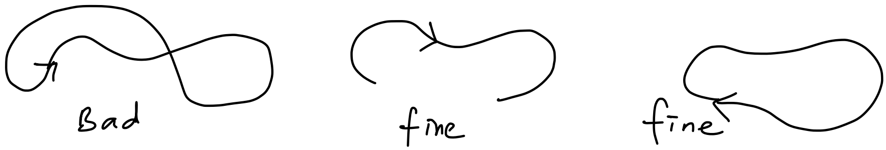
$$= \int_0^{2\pi} 1 \, dt$$

$$= \boxed{2\pi}$$

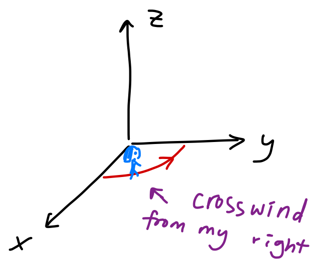
Part III . Flux

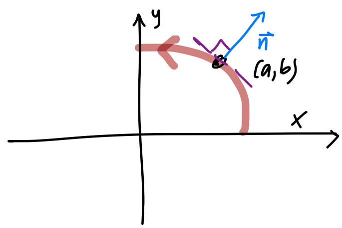
Idea : vector field $\vec{F}$ is the wind pattern.

I walk along a path C (doesn't have to be closed, but it shouldn't intersect itself)



The flux of $\vec{F}$ across C is the net amount of crosswind from my left (positive contribution) and crosswind from my right (negative contribution).



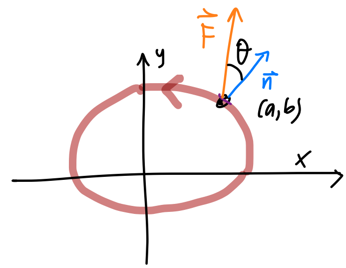


At each point (a,b) on C, there are two directions (on xy-plane) that are perpendicular to the line tangent to C at (a,b), that is, pointing away from my right shoulder and pointing away from my left shoulder.

choose this direction

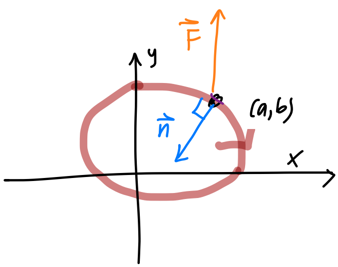
Let $\vec{n}$ be the unit vector pointing to the right of point (a,b) (when viewed from above).

Note: If $\vec{F}$ and $\vec{n}$ are both pointing "outward",



$\vec{F}\cdot\vec{n}$ is positive.

This is why crosswind from my left gives positive contribution.



If one of them is pointing outward and the other points inward, $\vec{F}\cdot\vec{n}$ is negative.

This is why crosswind from my right gives negative contribution

Def (Flux)

Let $\vec{F}=\langle f, g\rangle$ be a vector field of $\mathbb{R}^2$.

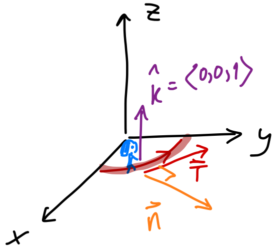
Let C be an oriented curve. Choose a parametrization of C: $\vec{r}(t)=\langle x(t), y(t)\rangle$, for $a\le t\le b$.

The <u>flux</u> of $\vec{F}$ across C is $\int_C \vec{F}\cdot\vec{n}\, ds$

Note:

If C is a closed curve with counterclockwise orientation, $\vec{n}$ is the unit normal vector pointing outward, so we say $\int_C \vec{F}\cdot\vec{n}\, ds$ gives the <u>outward flux across C.</u>

How to compute $\vec{n}$? $\quad \vec{n} = \vec{T} \times \hat{k}$



$$\vec{r} = \langle x(t), y(t), 0 \rangle, \qquad \vec{T} = \left\langle \underbrace{\frac{x'(t)}{|\vec{r}'(t)|}}_{T_1(t)}, \underbrace{\frac{y'(t)}{|\vec{r}'(t)|}}_{T_2(t)}, 0 \right\rangle$$

$$\vec{n} = \vec{T} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ T_1 & T_2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = T_2 \hat{i} - T_1 \hat{j} = \langle T_2, -T_1, 0 \rangle$$

$$= \frac{\langle y'(t), -x'(t), 0 \rangle}{|\vec{r}'(t)|}$$

How to compute the flux integral?

$$\int_C \vec{F} \cdot \vec{n} \, ds = \int_a^b \langle f, g \rangle \cdot \frac{\langle y'(t), x'(t) \rangle}{\cancel{|\vec{r}'(t)|}} \overbrace{\cancel{|\vec{r}'(t)|} \, dt}^{ds}$$

$$= \int_a^b \langle f, g \rangle \cdot \langle y'(t), -x'(t) \rangle \, dt$$

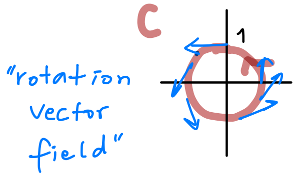
$$= \int_a^b f(t) \, y'(t) - g(t) \, x'(t) \, dt$$

Ex 8 (b)

Going back to Ex 6 (b)

$\vec{F}(x,y) = \langle \underset{f}{-y}, \underset{g}{x} \rangle$

C is counterclock unit circle



* Earlier, we computed line integral of $\vec{F}$ over C (circulation of $\vec{F}$ on C)

$$\int_C \vec{F} \cdot \vec{T}\, ds = 2\pi$$

The wind is always tailwind on my back, so the circulation is exactly the length of C

* Now, I expect $\int_C \vec{F} \cdot \vec{n}\, ds$ (called outward flux of $\vec{F}$ across C) to be 0 because there is no wind from my left or my right (all wind is tangent to C).

Do the actual computation:

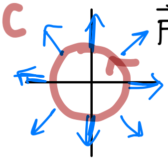
$\vec{r} = \langle \overbrace{\cos t}^{x}, \overbrace{\sin t}^{y} \rangle, \quad 0 \le t \le 2\pi$

$\vec{r}\,' = \langle \underbrace{-\sin t}_{x'}, \underbrace{\cos t}_{y'} \rangle$

$$\int_C \vec{F} \cdot \vec{n}\, ds = \int_a^b \underset{-y}{f}\, y' - \underset{x}{g}\, x'\, dt = \int_0^{2\pi} \overbrace{-(\sin t)(\cos t) - (\cos t)(-\sin t)}^{0}\, dt$$

$$= \int_0^{2\pi} 0\, dt = 0$$, as I expected.

For

Ex 6 (a) C  $\vec{F}(x,y) = \langle x, y \rangle$

I expect the outward flux $\int_C \vec{F} \cdot \vec{n}\, ds$ to be positive because $\vec{F}$ is always pointing in the same direction as $\vec{n}$.

Wind is always coming from my left (pos contribution).