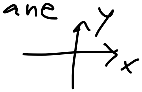
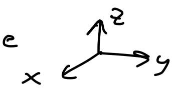
17.1 Vector fields

$\mathbb{R} = \{\text{all real numbers}\}$ — identify with point on real line $\longleftrightarrow$

$\mathbb{R}^2 = \{(x,y) : x, y \text{ are real numbers}\}$ — we can identify with points & vectors in xy-plane



$\mathbb{R}^3 = \{(x,y,z) : x,y,z \text{ are real numbers}\}$ — " — in xyz-space



Ch 14: A <u>vector-valued function</u> $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

or $\langle f(t), g(t) \rangle$

Ex: $\vec{r}(t) = \langle 5\cos t, 5\sin t \rangle$

domain (inputs) of $\vec{r}$: numbers in $\mathbb{R}$

range / image (outputs) of $\vec{r}$: vectors in $\mathbb{R}^3$ or $\mathbb{R}^2$

This vector symbol reminds you outputs are in $\mathbb{R}^2$ or $\mathbb{R}^3$

Ch 15:

A <u>scalar-valued</u> (or real-valued) <u>function of two or three variables</u> $F(x,y)$ or $F(x,y,z)$

Ex: $F(x,y) = 4e^{xy}$

domain of F : points in $\mathbb{R}^2$ or $\mathbb{R}^3$

range/image of F : numbers (i.e. scalar) in $\mathbb{R}$

Now Ch 17:

<u>Vector field</u> has domain in $\mathbb{R}^2$ and range/ image also in $\mathbb{R}^2$ (or both domain and range in $\mathbb{R}^3$)

Def A <u>vector field</u> in $\mathbb{R}^2$ is a function $\vec{F}$ written as

$$\vec{F}(x,y) = \langle f(x,y),\ g(x,y) \rangle \quad \text{or}$$

$$\vec{F}(x,y) = f(x,y)\,\hat{\imath} + g(x,y)\,\hat{\jmath}$$

- domain of $\vec{F}$: $R = \{ (x,y) : f(x,y)$ is defined, $g(x,y)$ is defined $\}$ in $\mathbb{R}^2$
- range / image of $\vec{F}$ is in $\mathbb{R}^2$

Say that $\vec{F}$ is continuous (differentiable) on a region R of $\mathbb{R}^2$ if f and g are both continuous (differentiable) on R.

<u>Note:</u>

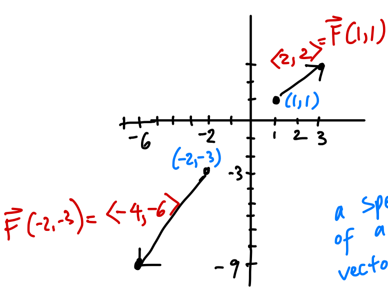
We can't visualize a vector field in its entirety, but we can plot a representative sample of vectors that illustrates the general appearance of the vector field.

Ex: Sketch the vector field

$$\vec{F}(x,y) = \langle 2x, 2y \rangle = 2x\,\hat{\imath} + 2y\,\hat{\jmath}$$

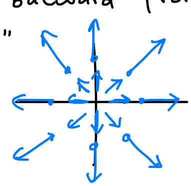
Sol: Select some points $P(a,b)$ in domain of $\vec{F}$, and plot a vector $\vec{F}(a,b)$ with tail at $P(a,b)$:

$P(a,b)$	$\vec{F}(a,b)$
$(1,1)$	$\langle 2,2 \rangle$
$(-2,-3)$	$\langle -4,-6 \rangle$
(a,b)	$\langle 2a, 2b \rangle$



Observe:
For each (a,b) except $(0,0)$, the vector $\vec{F}\langle a,b \rangle = \langle 2a, 2b \rangle$ points "outward from the origin"

a special case of a radial vector field



Ex 1 (b) Sketch the vector field $\vec{F}(x,y) = \langle 1-y^2, 0 \rangle$ for $|y| \le 1$

Sol: The domain of $\vec{F}$ is $\{(x,y) : x$ any real numbers, $-1 \le y \le 1\}$.

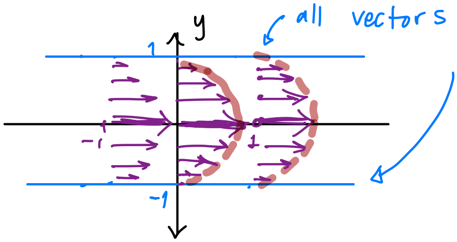
Because there is no "x" in the formula for $\vec{F}$, this vector field is independent of x. This means if you shift your point to the left or right (without changing the height), your vector doesn't change.

$(0,0)$ $\qquad\qquad$ $(5,0)$

$\vec{F}(0,0) = \langle 1, 0 \rangle$ $\qquad\qquad$ $\vec{F}(5,0) = \langle 1, 0 \rangle$

Since $1-y^2 > 0$ for $|y| < 1$, all vectors point in the positive x-direction in this region

all vectors are 0 at the boundaries $y=1$ and $y=-1$



This vector field might model the flow of water in a channel.

Gradient fields & potential functions

One way to generate a vector field is to

① start with a differentiable scalar-valued function in two variables $\varphi(x,y)$, (Ch 15)

② take its gradient $\nabla\varphi(x,y)$

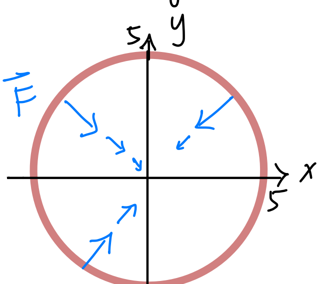
③ Define $\vec{F} = \nabla\varphi$.

A vector field $\vec{F}$ defined this way is called a <u>gradient field</u>, and φ is called a <u>potential function</u>

Ex 4: $\varphi(x,y) = 200 - x^2 - y^2$ with domain $R = \{(x,y) : x^2+y^2 \le 25\}$

Then the gradient field $\vec{F}$ associated with potential function φ is

$$\vec{F}(x,y) = \langle \varphi_x, \varphi_y \rangle = \langle -2x, -2y \rangle$$



$$|\vec{F}| = \sqrt{(-2x)^2 + (-2y)^2} = \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2}$$

At the boundary, $|\vec{F}| = 2\sqrt{25} = 10$.

$|\vec{F}|$ decreases toward the center of the disk.

Note: If $\varphi(x,y)$ is a temperature function, then the gradient field $\vec{F}$ gives, at each point, the direction in which the temp is increasing most rapidly. The magnitude of the vector is the rate of the increase.

Note: Disk is coolest on the boundary.
Disk is hottest (200°) at the center.

Given a scalar-valued (potential) function $\varphi(x,y)$, consider the surface $z = \varphi(x,y)$. Back in Ch 15, we can represent this surface by <u>level curves</u> in the xy-plane

A <u>level curve</u> of φ is the (2D) curve $z_0 = \varphi(x,y)$ in the xy-plane where z_0 is in the image/ range of φ.

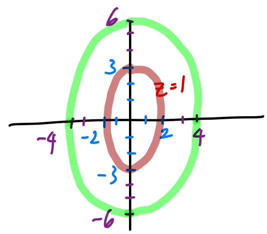
Ex: $\varphi(x,y) = \frac{x^2}{2^2} + \frac{y^2}{3^2}$ Range of $\varphi = [0, \infty)$

(meaning z_0 must be 0 or bigger)

Level curve for $z_0 = 1$: $1 = \frac{x^2}{2^2} + \frac{y^2}{3^2}$

Level curve for $z_0 = 4$: $4 = \frac{x^2}{4} + \frac{y^2}{9}$

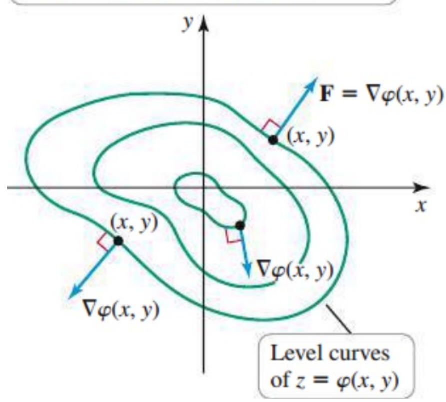
$\iff 1 = \frac{x^2}{16} + \frac{y^2}{36}$



<u>New Def</u> The <u>equipotential curves</u> of a potential function $\varphi(x,y)$ are the level curves of φ.

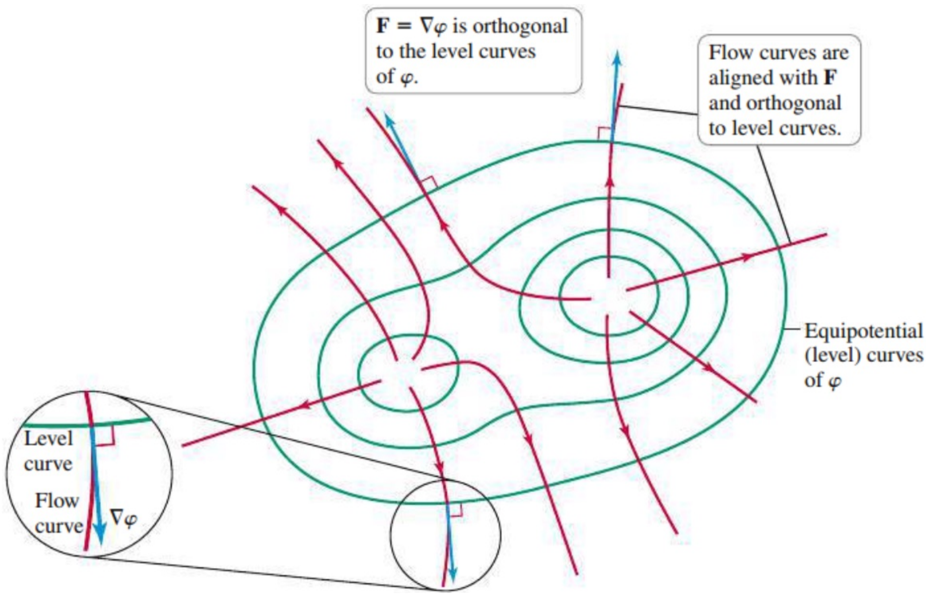
$\boxed{\text{Fact}}$ At each point (a,b) on a level curve of φ, the gradient $\nabla\varphi(a,b)$ is orthogonal (perpendicular) to the level curve

The vector field $\mathbf{F} = \nabla\varphi$ is orthogonal to the level curves of φ at (x, y).



This also means... The (gradient) vector field $\vec{F} = \nabla\varphi$ is orthogonal to the equipotential curves everywhere.

Cartoon:



Procedure for finding where $\vec{F}$ is normal or tangent to a curve:

Consider a vector field $\vec{F}$ in $\mathbb{R}^2$ and a curve C in the xy-plane. How to find where $\vec{F}$ is normal or tangent to C:

1. Think of C as a level curve $z_0 = \varphi(x,y)$ in the xy-plane of a surface $z = \varphi(x,y)$

2. Compute the gradient $\nabla\varphi(x,y)$.

3. Note to self: At each point (a,b) on the curve C, the gradient vector $\nabla\varphi(a,b)$ is orthogonal/perpendicular to the line tangent to C at (a,b).

If your goal is to find where $\vec{F}$ is normal to C:

4. Find (a,b) such that $\underbrace{\vec{F}(a,b) = k\,\nabla\varphi(a,b)}$ for nonzero scalar k

   (means vector $\vec{F}(a,b)$ is parallel to $\nabla\varphi(a,b)$)

   Then substitute into C to solve for a,b.

If your goal is to find where $\vec{F}$ is tangent to C:

5. Find (a,b) such that $\underbrace{\vec{F}(a,b) \cdot \nabla\varphi(a,b) = 0}$

   (means vector $\vec{F}(a,b)$ is perpendicular to $\nabla\varphi(a,b)$, meaning $\vec{F}(a,b)$ is parallel to tangent line to C at (a,b).)

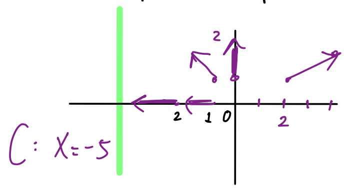
   Then substitute into C to solve for a,b.

Ex 2 / MML #5

Consider the vector field $\vec{F}(x,y) = \langle x, y \rangle$ and the curve $C = \{(x,y) : x = -5\}$

i) sketch C and a few representative vectors of $\vec{F}$.

Sol:



ii) Find points on curve C at which the vector field $\vec{F}$ is normal to C

↳ meaning the vector of $\vec{F}(a,b)$ at (a,b) is orthogonal/perpendicular to the line tangent to C at (a,b).

Sol: In this simple example, I see this happens $\boxed{\text{at point } (-5,0)}$

$\longleftarrow$ $\vec{F}(-5,0)$ is perpendicular to C

I can also find this point using the procedure from above:

To find where $\vec{F}$ is normal to C:

1. Think of C as a level curve $z_0 = \varphi(x,y)$ of a surface $z = \varphi(x,y)$

   $-5 = x$ $\qquad$ $z = x$

   So $\varphi(x,y) = x$

2. Compute $\nabla\varphi(x,y) = \langle 1, 0 \rangle$

3. Note to self: At each point (a,b) on C, $\nabla\varphi(a,b) = \langle 1,0 \rangle$ is orthogonal to the line tangent to C at (a,b).

4. Find (a,b) such that $\underbrace{\vec{F}(a,b) = k\,\nabla\varphi(a,b)}$ for scalar $k \neq 0$

   means vector $\vec{F}(a,b)$ is parallel to $\nabla\varphi(a,b)$

   Set $\langle a,b \rangle = k\langle 1,0 \rangle \Rightarrow \boxed{a = k \text{ and } b = 0}$

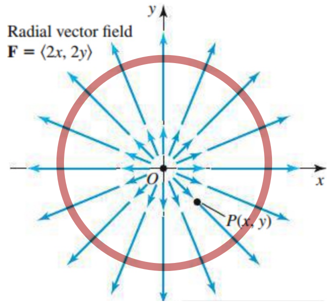
   Subs into C: $k = -5$

   So the answer is at point $\boxed{(-5,0)}$.

Ex 2 / MML#4 (Additional Example)

Consider the vector field $\vec{F}(x,y) = \langle 2x, 2y \rangle$
and the curve $C = \{(x,y) : x^2 + y^2 = 9\}$

i) Sketch C and a few representative vectors of $\vec{F}$.



C

ii) At which points of the curve C is $\vec{F}$ tangent to C?

From the sketch, we can guess the answer is:
The vector field is tangent to C at no points.

To find where $\vec{F}$ is tangent to C:

1. Think of C as a level curve $z_0 = \varphi(x,y)$ of a surface
$9 = x^2 + y^2$ $z = \varphi(x,y)$, $z = x^2 + y^2$

So $\varphi(x,y) = x^2 + y^2$

2. Compute $\nabla\varphi(x,y) = \langle 2x, 2y \rangle$

3. Note to self: At each point (a,b) on C, $\nabla\varphi(a,b) = \langle 2a, 2b \rangle$ is orthogonal to the line tangent to C at (a,b).

5. Solve $\vec{F}(a,b) \cdot \nabla\varphi(a,b) = 0$

$\langle 2a, 2b \rangle \cdot \langle 2a, 2b \rangle = 0 \Rightarrow 4a^2 + 4b^2 = 0 \Rightarrow a = 0,\ b = 0$

The point $(0,0)$ is not on C, so there are no points where $\vec{F}$ is tangent to C.