16.5 Triple integrals in (I) cylindrical and (II) spherical coordinates

polar coordinates in $\mathbb{R}^2$: r, θ
(the xy-plane)

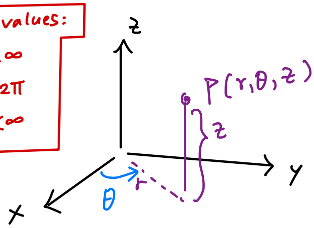
I cylindrical coordinates r, θ, z
polar coordinates in xy-plane; usual

II spherical coordinates $\rho\ \cancel{r}, \theta, \varphi$
rho
(counterclock) angle from positive x-axis
(~~counterclock~~) angle from positive z-axis

cylindrical coordinates $\underbrace{r, \theta}_{\text{same as before}}, z$

z = vertical position of the point
When P is projected onto the xy-plane,
r = distance from the origin
θ = angle (counterclock) from the positive x-axis

$\boxed{\text{possible values: } 0 \le r < \infty,\quad 0 \le \theta \le 2\pi,\quad -\infty < z < \infty}$



Transformation formulas (same as Sec 16.3)

Rectangular $(x, y, z) \to$ Cylindrical (r, θ, z)

$r^2 = x^2 + y^2$

$\sin\theta = \frac{x}{r}$ or $\cos\theta = \frac{y}{r}$ or $\tan\theta = \frac{y}{x}$

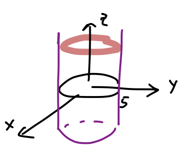
$z = z$

Cylindrical $(r, \theta, z) \to$ Rectangular (x, y, z)

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

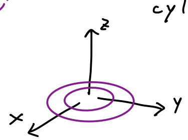
Ex of solids:

$D = \{(r, \theta, z) : r = 5\}$

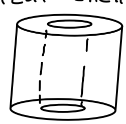


cylinder

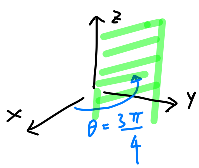
$D = \{(r, \theta, z) : 3 \le r \le 5\}$



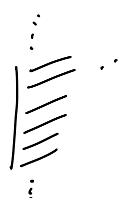
cylindrical shell



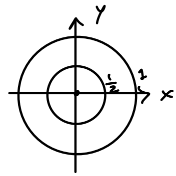
$D = \{(r, \theta, z) : \theta = \frac{3\pi}{4}\}$

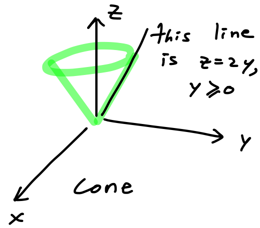


vertical half-plane



$D = \{(r,\theta,z) : z = 2r\}$





At $z=0$, the level curve is $r = 0$ (the origin)

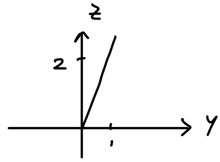
At $z=1$, the level curve is $1 = 2r \iff r = \frac{1}{2}$

At $z=2$, — " — $2 = 2r \iff r = 1$

The yz-trace is when $x=0$: $z = 2r$

$z = 2\sqrt{x^2+y^2}$

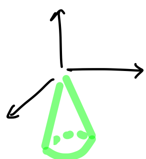
$z = 2y$



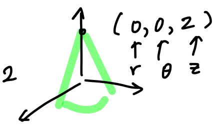
$D = \{(r,\theta,z) : z = 2-2r,\ 0 \leq r \leq 1\}$

The same cone as above but flipped and shifted up by 2

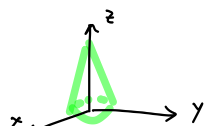
flipped:



shifted up by 2



cut off when $r = 1 \iff z = 2-2(1) = 0$:



Thm (Cylindrical)

If $R = \{(r,\theta) : g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta\}$ (described in polar) in the xy-plane and

D is the region in $\mathbb{R}^3$ bounded by surfaces $z = G(x,y)$ and $z = H(x,y)$,

$$\iiint_D f(x,y,z)\, dV = \iint_R \left[\int_{G(x,y)}^{H(x,y)} f(x,y,z)\, dz \right] dA$$

i.e. If $D = \{(r,\theta,z) : g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta,\ G(x,y) \leq z \leq H(x,y)\}$,

$$\iiint_D f(x,y,z)\, dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \left[\int_{G(r\cos\theta,\, r\sin\theta)}^{H(r\cos\theta,\, r\sin\theta)} f(r\cos\theta,\ r\sin\theta,\ z)\, dz \right] r\, dr\, d\theta$$

multiply by r

Ex 2 (book)

Evaluate $I = \int_0^{2\sqrt{2}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \int_{-1}^{2} \underbrace{\sqrt{1+x^2+y^2}}_{f(x,y,z)} \, dz \, dy \, dx$

Convert the triple integral to an equivalent triple integral in cylindrical coordinates

Sol: This is $\iiint_D f(x,y,z) \, dV$ where

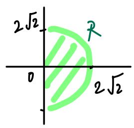
$$D = \{ (x,y,z) : -1 \le z \le 2, \quad \text{(inner)}$$
$$-\sqrt{8-x^2} \le y \le \sqrt{8-x^2}, \quad \text{(middle)}$$
$$0 \le x \le 2\sqrt{2} \} \quad \text{(outer)}$$

the projection of D onto the xy-plane is $R = \{(x,y): -\sqrt{8-x^2} \le y \le \sqrt{8-x^2}, \ 0 \le x \le 2\sqrt{2} \}$

y is bounded above by $y = \sqrt{8-x^2}$ (upper half of circle)

$y^2 = 8 - x^2$

$x^2 + y^2 = 8$ (circle w/ radius $\sqrt{8}$)



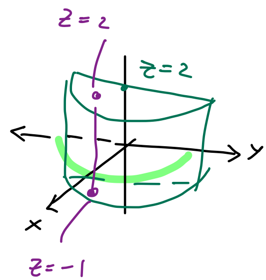
In polar, $R = \{(r,\theta): 0 \le r \le 2\sqrt{2}, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \}$

right half of a disk

In cylindrical,

$D = \{ (r,\theta,z): 0 \le r \le 2\sqrt{2}, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, -1 \le z \le 2 \}$

half of a solid cylinder



$f(x,y,z) = \sqrt{1 + \underbrace{x^2+y^2}_{r^2}} = \sqrt{1+r^2}$

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\sqrt{2}} \left[\int_{-1}^{2} \sqrt{1+r^2} \, dz \right] r \, dr \, d\theta$

extra

inner $\int_{-1}^{2} \sqrt{1+r^2} \, dz = \sqrt{1+r^2} \int_{-1}^{2} dz = 3\sqrt{1+r^2}$

middle $\int_0^{2\sqrt{2}} 3\sqrt{1+r^2} \, r \, dr = 3 \int_{u=1+0}^{u=1+8} \sqrt{u} \, \frac{1}{2} du = \frac{3}{2} \left. \frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right|_1^9$

extra

$u = 1 + r^2$

$du = 2r \, dr$

$\frac{1}{2} du = r \, dr$

$= 9^{\frac{3}{2}} - 1$

$= 3^3 - 1$

$= 26$

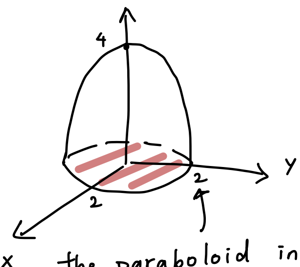
outer $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 26 \, d\theta = 26\pi$

$\boxed{I = 26\pi}$

(Additional example)

Ex 3: Find the mass of the solid D bounded by the paraboloid $z = 4 - r^2$ and the plane $z = 0$. The density of the solid is $f(r,\theta,z) = 5 - z$ (heavy near the base and light near the top).

Sol: Sketch D:



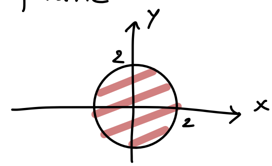
the paraboloid intersects the xy-plane when

$0 = 4 - r^2$

$r^2 = 4$

$r = 2$ (circle w/ radius 2)

The projection R of D onto the xy-plane



is $R \overset{\text{in polar}}{=} \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

so $D \underset{\text{in cylindrical}}{=} \{(r,\theta,z) : \quad ,\ 0 \le z \le 4 - r^2\}$

The mass of D is

$$\underset{D}{\iiint} \underset{\substack{\text{density}\\\text{function}}}{f}\, dV = \int_0^{2\pi}\int_0^2 \underbrace{\int_0^{4-r^2} (5-z)\,dz}_{\text{inner}}\ \overset{\boxed{\text{extra}}}{r}\,dr\,d\theta$$

$\underline{\text{inner}}$ $\displaystyle\int_0^{4-r^2} 5 - z\,dz = 5z - \frac{z^2}{2}\Bigg|_{z=0}^{z=4-r^2}$

$= 5(4-r^2) - \dfrac{(4-r^2)^2}{2}$

$= 20 - 5r^2 - \frac{1}{2}(16 - 8r^2 + r^4)$

$= 12 - r^2 - \frac{1}{2}r^4$

$\underline{\text{middle}}$ $\displaystyle\int_0^2 12r - r^3 - \frac{1}{2}r^5\,dr$

$= \dfrac{12r^2}{2} - \dfrac{r^4}{4} - \dfrac{1}{2}\dfrac{r^6}{6}\Bigg|_{r=0}^{r=2}$

$= 6(2)^2 - \dfrac{2^4}{4} - \dfrac{1}{2}\dfrac{2^6}{6}$

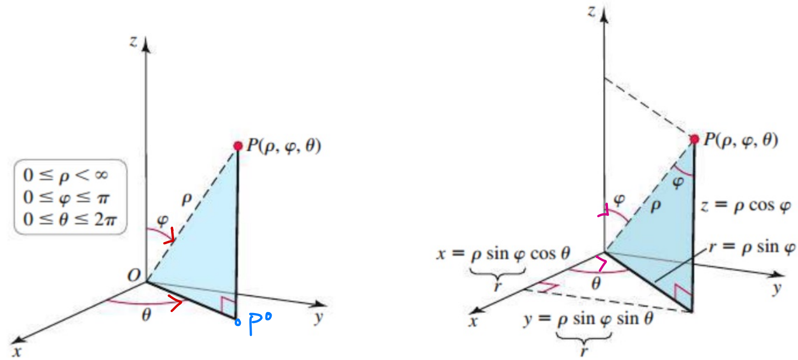
$= 24 - 4 - \dfrac{16}{3} = \dfrac{44}{3}$

$\underline{\text{outer}}$ $\displaystyle\int_0^{2\pi} \frac{44}{3}\,d\theta = \frac{44}{3}(2\pi) = \boxed{\frac{88\pi}{3}}$

The mass

II Spherical coordinates (ρ, φ, θ)

rho phi theta (same as in cylindrical)



Here P° is the projection of P onto the xy-plane

r is the distance between O and P°.

ρ: distance between the origin and the point P, ρ in $[0, \infty)$

φ: angle between the positive z-axis and the line OP (origin, point)

θ: same as in polar & cylindrical coordinates:
(counterclock) rotation from positive x-axis to the line OP° on the xy-plane

Transformations

Spherical $(\rho, \varphi, \theta) \to$ Rectangular (x, y, z)

$\cos \varphi = \frac{\text{adj}}{\text{hyp}} = \frac{z}{\rho}$ so $z = \rho \cos \varphi$

$\sin \varphi = \frac{\text{opp}}{\text{hyp}} = \frac{r}{\rho}$ and $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$ so $x = \overbrace{\rho \sin \varphi}^{r} \cos \theta$, $y = \underbrace{\rho \sin \varphi}_{r} \sin \theta$

$r = \rho \sin \varphi$

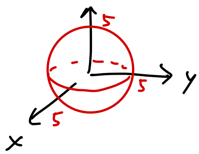
Transformations Between Spherical and Rectangular Coordinates

Rectangular $\to$ Spherical	Spherical $\to$ Rectangular
$\rho^2 = x^2 + y^2 + z^2$	$x = \rho \sin \varphi \cos \theta$
Use trigonometry to find	$y = \rho \sin \varphi \sin \theta$
φ and θ.	$z = \rho \cos \varphi$

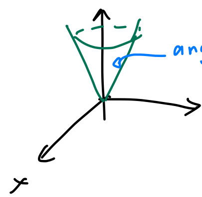
Examples:

- Sphere centered at the origin with radius 5:

$$\{(\rho, \varphi, \theta) : \rho = 5\}$$



- Cone $\{(\rho, \varphi, \theta) : \varphi = \frac{\pi}{6}\}$



angle between z-axis and cone is $\frac{\pi}{6}$

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Ex 5 (a)

Convert to rectangular coordinates & identify the set:

$$D = \{(\rho, \varphi, \theta) : \rho = 2\cos\varphi,\ 0 \le \varphi \le \tfrac{\pi}{2},\ 0 \le \theta \le 2\pi\}$$

Sol: (To avoid working w/ square roots, multiply both sides by ρ)

$$\underbrace{\rho^2}_{x^2+y^2+z^2} = 2\underbrace{\rho\cos\varphi}_{z}$$

$$x^2 + y^2 + z^2 = 2z$$

$$x^2 + y^2 + z^2 - 2z = 0$$

Complete the square: $\left(z^2 - 2z + 1^2 = (z-1)^2\right)$

$$x^2 + y^2 + (z-1)^2 = 1$$

Sphere centered at $(0,0,1)$ w/ radius 1.

THEOREM 16.7 Change of Variables for Triple Integrals in Spherical Coordinates

Let f be continuous over the region D, expressed in spherical coordinates as

$$D = \{(\rho, \varphi, \theta): 0 \le g(\varphi, \theta) \le \rho \le h(\varphi, \theta), a \le \varphi \le b, \alpha \le \theta \le \beta\}.$$

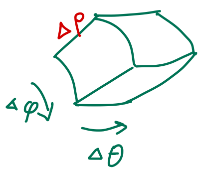
Then f is integrable over D, and the triple integral of f over D is

$$\iiint_D f(x, y, z)\, dV$$
$$= \int_\alpha^\beta \int_a^b \left[\int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho\cos\varphi)\overbrace{\rho^2 \sin\varphi}^{\text{extra}}\, d\rho\right] d\varphi\, d\theta.$$

Idea:

We replace $dx\,dy\,dz$ with $\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$ because

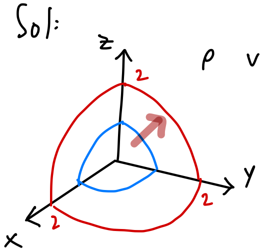
volume of  is $(\Delta x)(\Delta y)(\Delta z)$ vs

volume of 

is $(\Delta\rho)^2 (\sin\varphi)(\Delta\rho)(\Delta\varphi)(\Delta\theta)$

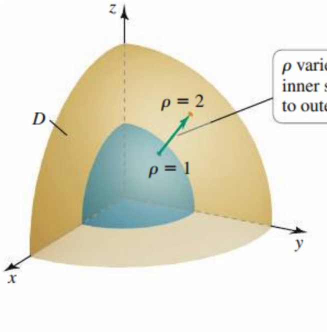
Ex 6: Let D be the region in the first octant between two spheres of radius 1 and 2 centered at the origin. Evaluate $\iiint_D (x^2+y^2+z^2)^{-\frac{3}{2}}\, dV$.

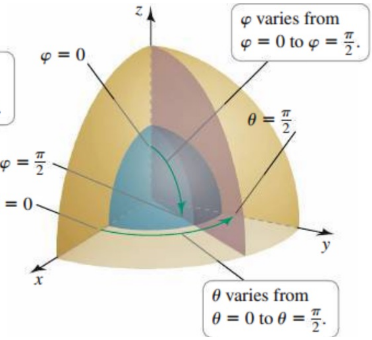
Sol:



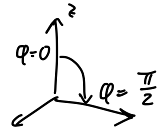
ρ varies from inner sphere $(\rho = 1)$ to outer sphere $(\rho = 2)$

So $1 \le \rho \le 2$



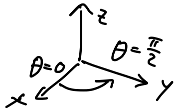


φ varies from $\varphi = 0$ to $\varphi = \frac{\pi}{2}$



So $0 \le \varphi \le \frac{\pi}{2}$

θ varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$



So $D = \{ (\rho, \varphi, \theta) : 1 \le \rho \le 2,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le \frac{\pi}{2} \}$

$$f(x,y,z) = (x^2 + y^2 + z^2)^{-\frac{3}{2}} = (\rho^2)^{-\frac{3}{2}} = \rho^{-3}$$

$$\iiint_D f \, dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_1^2 \rho^{-3} \overbrace{\rho^2 \sin\varphi}^{\text{extra}} \, d\rho \, d\varphi \, d\theta$$

$$= \int \int \underbrace{\int \rho^{-1} \, d\rho}_{\text{inner}} \sin\varphi \, d\varphi \, d\theta$$

<u>inner</u> $\int_1^2 \rho^{-1} \, d\rho = \ln|\rho| \Big|_{\rho=1}^{\rho=2} = \ln(2) - \ln(1) = \ln(2).$

<u>middle</u> $\int_0^{\frac{\pi}{2}} \ln(2) \sin\varphi \, d\varphi = \ln(2) \left(-\cos\varphi\right) \Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}}$

$$= -\ln(2) \left[\cos\tfrac{\pi}{2} - \cos 0 \right]$$

$$= \ln(2)$$

<u>outer</u> $\int_0^{\frac{\pi}{2}} \ln(2) \, d\theta = \boxed{\ln(2) \, \frac{\pi}{2} = \iiint_D (x^2 + y^2 + z^2)^{-\frac{3}{2}} \, dV.}$