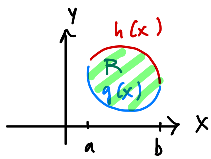
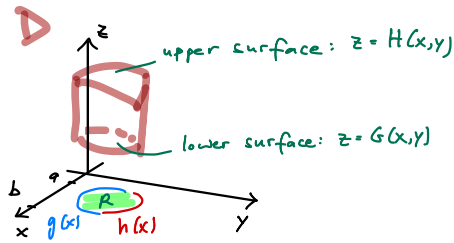
16.4 Triple integrals

Let D be a region in $\mathbb{R}^3$

$D = \{(x,y,z) : a \le x \le b,\ g(x) \le y \le h(x),\ G(x,y) \le z \le H(x,y)\}$





Let $f(x,y,z)$ be a function which is continuous over the region D.

The <u>triple integral of f over D</u> $\displaystyle\iiint_D f(x,y,z)\,dV$

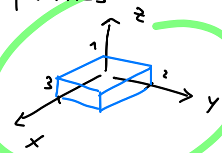
can be computed as the iterated integral $\displaystyle\iint_R \left[\int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dz\right] dA$

or $\displaystyle\int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dz\,dy\,dx$

$$D = \{(x,y,z) : \underbrace{a \le x \le b}_{\text{outer integral}},\ \underbrace{g(x) \le y \le h(x)}_{\text{middle integral}},\ \underbrace{G(x,y) \le z \le H(x,y)}_{\text{inner integral}}\}$$

Ex 1 (Mass of a box) A solid box D is bounded by the planes

$x=0$, $x=3$, $y=0$, $y=2$, $z=0$, $z=1$.



The density of the box is given by $f(x,y,z) = 2-z$.

(The density decreases in the positive z direction.)

Then the mass of the box is $M = \iiint\limits_D f(x,y,z)\, dV$.

(Because the limits of integration for all three variables x, y, z are constant, the iterated integral may be written in any order.)

$$M = \int_0^3 \int_0^2 \left[\int_0^1 (2-z)\, dz \right] dy\, dx$$

$$= \int_0^3 \int_0^2 \left. 2z - \frac{z^2}{2} \right]_0^1 dy\, dx$$

$2 - \frac{1}{2} = \frac{3}{2}$

$$= \int_0^3 \int_0^2 \frac{3}{2}\, dy\, dx$$

$$= \int_0^3 \left. \frac{3}{2} y \right|_0^2 dx$$

$$= \int_0^3 3\, dx$$

$$= \left. 3x \right|_0^3 = 9$$

inner $\int_0^1 (2-z)\, dz$

$= \left. 2z - \frac{z^2}{2} \right]_0^1$

$= 2 - \frac{1}{2} = \frac{3}{2}$

middle $\int_0^2 \frac{3}{2}\, dy = \left. \frac{3}{2} y \right|_0^2 = 3$

outer $\int_0^3 3\, dx = \left. 3x \right|_0^3 = 9$

Note: Any other order of integration produces the same result.

Confidence check: At the top (bottom) of the box, density is $f(x,y,1) = 2-1 = 1$. $f(x,y,0) = 2-0 = 2$

If the box has constant density of 1, its mass would be (vol)(density) $= 6 \cdot 1 = 6$.

— " — (2) — " — $= 6 \cdot 2 = 12$

Actual mass is $\frac{6+12}{2} = \frac{18}{2} = 9$, as you might expect.

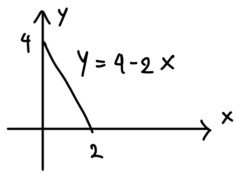
Ex 2: Find the volume of the prism D in the positive octant bounded by the planes $y = 4 - 2x$ and $z = 6$.

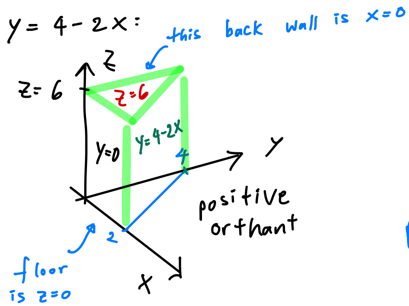
Sol: Volume is $V = \iiint_D 1 \, dV$

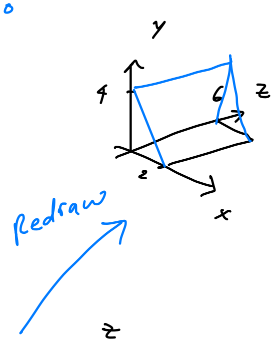
Sketch D.

The other bounds are the coordinate planes $x = 0$, $y = 0$, $z = 0$.

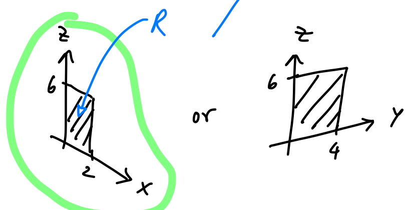
Sketch the xy-trace of $y = 4 - 2x$:







We can think of the rectangle as the base of the prism.



Let this be the base

Base of prism: $y = 0$ (the xz-plane)

Then the upper surface is $y = 4 - 2x$.

This means inner integral will be dy

Projection of prism onto the xz plane is

$$R = \{(x, z) : 0 \le x \le 2, \ 0 \le z \le 6\}$$

$$V = \iint_R \underbrace{\left(\int_0^{4-2x} 1 \, dy \right)}_{\text{inner}} dA = \int_0^6 \int_0^2 \underbrace{(4 - 2x)}_{\text{inner}} \, dx \, dz$$

$$= \iint_R \left(y \Big|_{y=0}^{y=4-2x} \right) dA \qquad = \int_0^6 4x - \frac{\cancel{2}x^2}{\cancel{2}} \Big|_{x=0}^{x=2} dz$$

$$= \iint_R (4 - 2x) \, dA \qquad = \int_0^6 \underbrace{(8 - 4)}_{\text{middle}} dz = 4(6) = 24$$

We can also do dy dz dx:

$$\int_0^2\int_0^6 (4-2x)\,dz\,dx = \int_0^2 (4-2x)z\Big|_{z=0}^{z=6}\,dx$$

Confidence check: Check that the result is also 24

$$= \int_0^2 24-12x\,dx$$

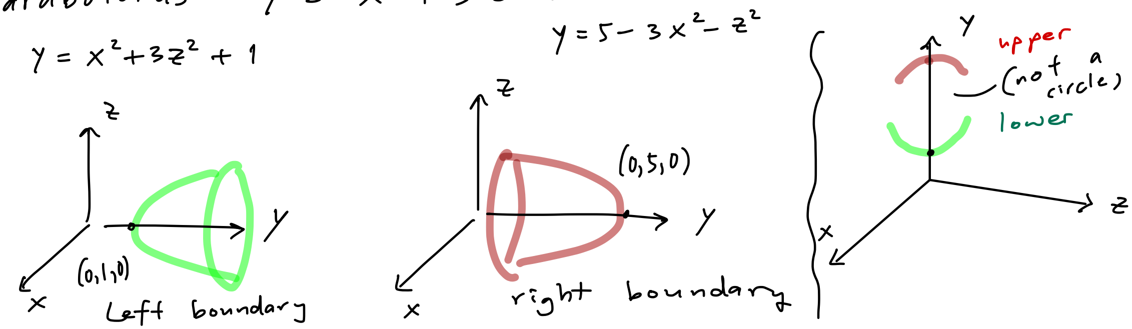
$$= 24x - \frac{12x^2}{2}\Big|_{x=0}^{x=2}$$

$$= 24(2) - 6(4) = 24$$

(MML #6)

Ex 3:

Find the volume of the solid D bounded by paraboloids $y = x^2+3z^2+1$ and $y=5-3x^2-z^2$



Find the intersection of $y=x^2+3z^2+1$ & $y=5-3x^2-z^2$ then project it onto the xz-plane:

Set
$$x^2+3z^2+1 = 5-3x^2-z^2$$
$$4x^2+4z^2 = 4$$
$$x^2+z^2 = 1 \quad (\text{unit circle on the } xz\text{-plane})$$

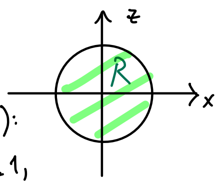
$$V = \iiint_D 1 \, dA = \iint_R \left[\int_{x^2+3z^2+1 \text{ (left boundary)}}^{5-3x^2-z^2 \text{ (right boundary)}} 1 \, dy \right] dA$$

where R is the disk $x^2+z^2 \leq 1$

inner $\displaystyle y \Big|_{\text{left bound}}^{\text{right bound}} = 5-3x^2-z^2-(x^2+3z^2+1)$

$$V = \iint_R 4(1-x^2-z^2) \, dA$$

Because R is a disk, it's easier to evaluate this double integral in polar coordinates (previous sec 16.3).
But now "z" plays the role of "y"



$R = \{(r,\theta) : 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi\}$

$$\iint_R f(x,z) \, dA = \int_0^{2\pi}\int_0^1 f(r\cos\theta, r\sin\theta) \, r \, dr$$

$$f(x,z) = 4(1-x^2-z^2)$$
$$= 4(1-r^2\cos^2\theta - r^2\sin^2\theta)$$
$$= 4(1-r^2) \quad \text{because} \quad \cos^2\theta + \sin^2\theta = 1$$

Alternatively, $f(x,z) = 4(1-(x^2+z^2)) = 4(1-r^2)$ because $x^2+z^2=r^2$

normally, $x^2+y^2=r^2$

$$\int_0^{2\pi}\int_0^1 4(1-r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 4(r-r^3) \, dr \, d\theta = \int_0^{2\pi} 4\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_{r=0}^{r=1} d\theta = \int_0^{2\pi} 1 \, d\theta = \boxed{2\pi}$$

Def ~ Average value of $f(x,y,z)$ over a region D (in $\mathbb{R}^3$) is

$$\frac{1}{\text{volume of } D} \iiint_D f(x,y,z)\, dV$$

Students try:

EXAMPLE 5 **Average temperature** Consider a block of a conducting material occupying the region

$$D = \{(x, y, z): 0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 1\}.$$

Due to heat sources on its boundaries, the temperature in the block is given by $T(x, y, z) = 250xy \sin \pi z$. Find the average temperature of the block.

SOLUTION We must integrate the temperature function over the block and divide by the volume of the block, which is 4. One way to evaluate the temperature integral is as follows:

$$\iiint_D 250xy \sin \pi z \, dV = 250 \int_0^2 \int_0^2 \int_0^1 xy \sin \pi z \, dz\, dy\, dx \qquad \text{Convert to an iterated integral.}$$

$$= 250 \int_0^2 \int_0^2 xy \frac{1}{\pi}(-\cos \pi z)\Big|_0^1 dy\, dx \qquad \text{Evaluate inner integral with respect to } z.$$

$$= \frac{500}{\pi} \int_0^2 \int_0^2 xy \, dy\, dx \qquad \text{Simplify.}$$

$$= \frac{500}{\pi} \int_0^2 x\left(\frac{y^2}{2}\right)\Big|_0^2 dx \qquad \text{Evaluate middle integral with respect to } y.$$

$$= \frac{1000}{\pi} \int_0^2 x\, dx \qquad \text{Simplify.}$$

$$= \frac{1000}{\pi}\left(\frac{x^2}{2}\right)\Big|_0^2 = \frac{2000}{\pi}. \qquad \text{Evaluate outer integral with respect to } x.$$

Dividing by the volume of the region, we find that the average temperature is $(2000/\pi)/4 = 500/\pi$.