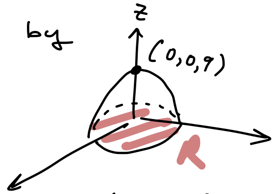
16.3 Double integrals in polar coordinates

Ex 1: Find the volume of the solid bounded by



the paraboloid $z = 9 - x^2 - y^2$

and the xy-plane $(z=0)$

How to Solve: The intersection of the paraboloid and xy-plane is found by setting the two surfaces equal:

$$0 = 9 - x^2 - y^2$$

$x^2 + y^2 = 9$ (circle w/ center at origin, w/ radius 3)

So $R = \{ (x,y) : x^2 + y^2 \leq 9 \}$

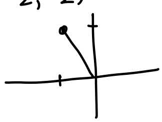
Volume is $V = \iint_R f(x,y)\, dA$ where $f(x,y) = 9 - x^2 - y^2$

(It's easier to compute V using polar coordinates!)

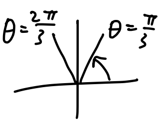
Recall dictionary for going between xy (Cartesian) coordinates & polar coordinates:

If I know x, y, to find r, θ, I use: $r^2 = x^2 + y^2$
and $\sin(\theta) = \frac{y}{r}$ or $\cos(\theta) = \frac{x}{r}$ or $\tan\theta = \frac{y}{x}$

Ex: $(x,y) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



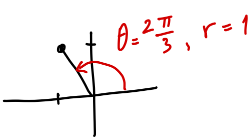
So $r^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$, so $\boxed{r = 1}$



$\sin\theta = \frac{y}{r} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{1}$ so $\theta = \frac{\pi}{3}$ or $\pi - \frac{\pi}{3}$

since my point is in the 2nd Quadrant,

I know $\boxed{\theta = \frac{2\pi}{3}}$ (not $\frac{\pi}{3}$)



$(x,y) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in Cartesian is the same as

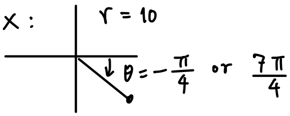
$(r,\theta) = \left(1, \frac{2\pi}{3}\right)$ in polar

$= \left(1, \frac{2\pi}{3} + 2\pi\right) = \left(1, \frac{2\pi}{3} + \underset{\text{even}}{14\pi}\right)$

$= \left(-1, \frac{2\pi}{3} + \pi\right) = \left(-1, \frac{2\pi}{3} + \underset{\text{odd}}{17\pi}\right)$

If I know r, θ, to find x, y, I use: $x = r\cos\theta$ and $y = r\sin\theta$

Ex:



$\Rightarrow$ $x = 5\cos\left(-\frac{\pi}{4}\right) = -5\frac{\sqrt{2}}{2}$, $y = 5\sin\left(-\frac{\pi}{4}\right) = -5\frac{\sqrt{2}}{2}$ $\Rightarrow$ So the point $\left(10, -\frac{\pi}{4}\right)$ in polar is $\left(-5\frac{\sqrt{2}}{2}, -5\frac{\sqrt{2}}{2}\right)$ in Cartesian

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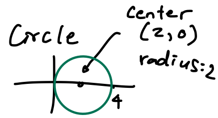
Ex: Given $r = 4\cos\theta$ in polar, convert to Cartesian:

$r^2 = 4r\cos\theta$

$x^2 + y^2 = 4x \Rightarrow x^2 - 4x + y^2 = 0 \Rightarrow$ (complete the square) $x^2 - 4x + 4 + y^2 = 4$

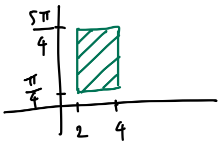
$(x-2)^2 + y^2 = 4$



Def A Cartesian rectangle is

$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

Ex: $R = \left\{(x,y) : 2 \le x \le 4,\ \frac{\pi}{4} \le y \le \frac{5\pi}{4}\right\}$

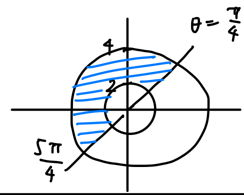


A polar rectangle is

$$R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$$

Ex: $R = \left\{(r,\theta) : 2 \le r \le 4,\ \frac{\pi}{4} \le \theta \le \frac{5\pi}{4}\right\}$

(MML #5)

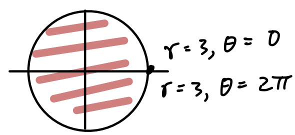


Another ex of a polar rectangle is

$$R = \{(r,\theta) : 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$$

= closed disk centered at the origin with radius 3

= the region we need to integrate over for our Ex 1.



Thm (Double Integrals over polar rectangles)

If $R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$

a nonnegative; $\beta - \alpha$ is not bigger than 2π

then
$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

extra

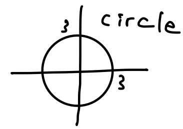
Sol of Ex 1:

The paraboloid $z = 9 - x^2 - y^2$ can be described in polar by

$$\boxed{z = 9 - r^2}$$

and xy-plane is still $\boxed{z = 0}$

Their intersection is when $0 = 9 - r^2 \iff r = 3$



So the region R is the disk (also a polar rectangle)

$R = \{ (r, \theta) : 0 \leq r \leq 3, \ 0 \leq \theta \leq 2\pi \}$

Volume of solid is $V = \iint_R \left(\underset{\text{paraboloid}}{\text{upper surface}} \right) - \left(\underset{\text{xy-plane}}{\text{lower surface}} \right) dA$

$$= \int_0^{2\pi} \int_0^3 \left((9 - r^2) - 0 \right) \overset{\text{extra}}{r} \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\overbrace{\int_0^3 9r - r^3 \, dr}^{\text{inner}} \right] d\theta$$

$$= \int_0^{2\pi} \frac{9r^2}{2} - \frac{r^4}{4} \Bigg|_{r=0}^{r=3} d\theta$$

$$= \int_0^{2\pi} \frac{9(9)}{2} - \frac{81}{4} \, d\theta$$

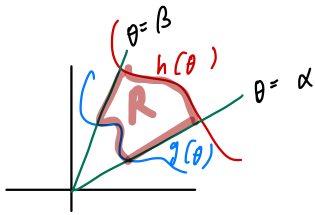
$$= \int_0^{2\pi} 81 \left(\frac{1}{2} - \frac{1}{4} \right) d\theta = \frac{81}{4} \theta \Big|_0^{2\pi} = \frac{81}{4} (2\pi) = \boxed{\frac{81}{2} \pi}$$

Next:
Like in Sec 16.2 (double integrals over non-rectangular region in Cartesian), we can have double integrals over regions R in polar where R is not a polar rectangle.

Thm (Double integral over general polar regions)

Let $R = \{ (r,\theta) : 0 \le \underset{\text{lower}}{g(\theta)} \le r \le \underset{\text{upper}}{h(\theta)},\ \alpha \le \theta \le \beta \}$

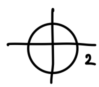
Difference is between 0 and 2π.



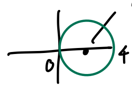
Then $$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta$$

Ex 4:
Describe in polar coordinates the region R in the xy-plane

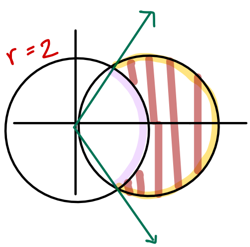
Outside the circle $r = 2$



and inside the circle $r = 4\cos\theta$



center (2,0)
previous example shows this was $x^2 + y^2 = 4x$



lower bound for r is $r = 2$
upper bound for r is $\boxed{r = 4\cos\theta}$

Bounds for θ are the values for θ when the two circles intersect:

Set $2 = 4\cos\theta$
$\frac{1}{2} = \cos\theta \Rightarrow \theta = \frac{\pi}{3}, -\frac{\pi}{3}$

$R = \{ (r,\theta) : 2 \le r \le 4\cos\theta,\ -\frac{\pi}{3} \le \theta \le \frac{\pi}{3} \}$

$$\boxed{\text{So } \iint_R f(r,\theta)\, dA = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_2^{4\cos\theta} f(r,\theta)\, r\, dr\, d\theta}$$

Def

(same def as before) (different computation)

Area of R is $A = \iint_R 1 \, dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} r \, dr \, d\theta$

In above example, area is $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{2}^{4\cos\theta} r \, dr \, d\theta$

Def

The average value of a function over a region R expressed in polar is (the same as in Sec 16.1)

$$\frac{1}{\text{area}(R)} \iint_R f(x,y) \, dA$$