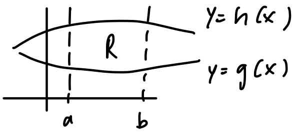
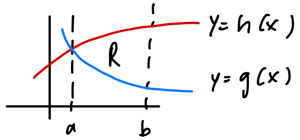
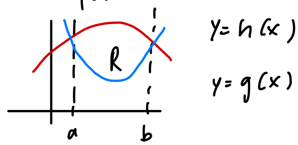
16.2 Double integrals over general regions

Consider a general (non-$\square$) region R w/ lower & upper boundaries $y=g(x), y=h(x)$ for $a \le x \le b$:







To find the volume of the solid bounded by the surface $z=f(x,y)$ and R, we compute double integral $\iint_R f(x,y)\,dA$ over R, using an iterated integral:

$$\iint_R f(x,y)\,dA = \int_a^b \Bigg(\underbrace{\int_{g(x)\,(\text{lower})}^{h(x)\,(\text{upper})} f(x,y)\,dy}_{A(x)} \Bigg)\,dx$$

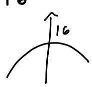
cross-sectional area of vertical slice parallel to the yz-plane (x-value is fixed)

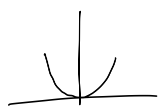
Ex 1: Let R be the region bounded by the parabolas $y=3x^2$ and $y=16-x^2$.

i) Express the double integral $\iint_R 2x^2y\,dA$ as an iterated integral

ii) Evaluate the integral

Sol: Sketch R:

$y=16-x^2$ 

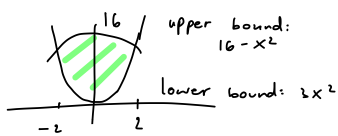
$y=3x^2$ 

To find the intersection points of these two curves, set

$3x^2 = 16-x^2$

$4x^2 = 16$

$x^2 = 4 \Rightarrow \boxed{x=\pm 2}$

R: 

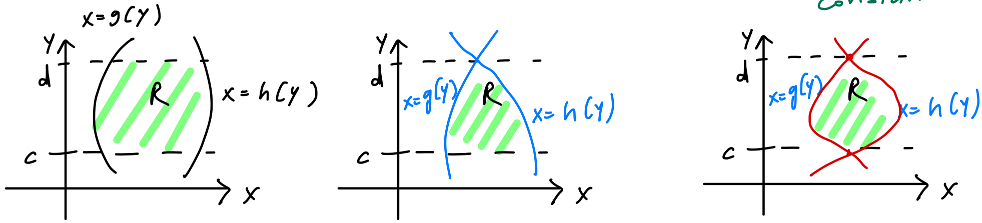
$$A(x) = \int_{3x^2}^{16-x^2} 2x^2y\,dy = 2x^2\frac{y^2}{2}\Bigg|_{y=3x^2}^{y=16-x^2} = x^2\left[(16-x^2)^2-(3x^2)^2\right]$$

$$= x^2(16^2-32x^2+x^4-9x^4) = x^2(256-32x^2-8x^4) = 256x^2-32x^4-8x^6$$

i) $$\iint_R f(x,y)\,dA = \int_{-2}^{2}\Bigg(\underbrace{\int_{3x^2}^{16-x^2} 2x^2y\,dy}_{A(x)}\Bigg)\,dx$$

ii) $$\iint_R f(x,y)\,dA = \int_{-2}^{2}\Big(\underbrace{256x^2-32x^4-8x^6}_{A(x)}\Big)\,dx = 256\frac{x^3}{3} - 32\frac{x^5}{5} - 8\frac{x^7}{7}\Bigg|_{x=-2}^{x=2}.$$

Now, consider a region R bounded on the left & right by $x = g(y)$, $x = h(y)$, on the interval $\underbrace{c \le y \le d}_{\text{constant bounds}}$.

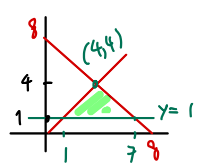


To find the volume of the solid bounded by the surface $z = f(x,y)$ and R, we take vertical slices parallel to the xz-plane (y-value is fixed).

$$\iint_R f(x,y)\, dA = \int_c^d \underbrace{\left(\int_{g(y)\,(\text{left})}^{h(y)\,(\text{right})} f(x,y)\, dx \right)}_{A(y)} dy$$

EX 2: Find the volume of the solid below the surface $f(x,y) = 2 + \frac{1}{y}$ and above the region R in the xy-plane bounded by the lines $y = x$, $y = 8 - x$, and $y = 1$. Note: $f(x,y) > 0$ on R.

Sketch R:



Left bound: $x = y$, right bound $x = 8 - y$,

so $A(y) = \displaystyle\int_y^{8-y} \left(2 + \frac{1}{y}\right) dx = \left(2 + \frac{1}{y}\right) x \Big|_{x=y}^{x=8-y} = \left(2 + \frac{1}{y}\right)(8 - y - y)$

$= \left(2 + \frac{1}{y}\right)(8 - 2y) = 16 - 4y + \frac{8}{y} - 2$

$= \boxed{14 - 4y + \frac{8}{y}}$

Volume is $\displaystyle\int_1^4 \underbrace{\int_y^{8-y} \left(2 + \frac{1}{y}\right) dx}_{A(y)}\, dy$

$= \displaystyle\int_1^4 14 - 4y + \frac{8}{y}\, dy$

$= 14y - \frac{4y^2}{2} + 8 \ln|y| \Big|_{y=1}^{y=4} = \boxed{12 + 8 \ln 4}$

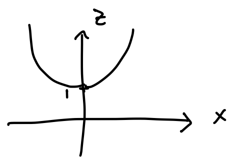
If x has non-constant bounds, do dx first

More general: The volume of the solid between surfaces $g(x,y)$ and $f(x,y)$ over a region R on the xy-plane is

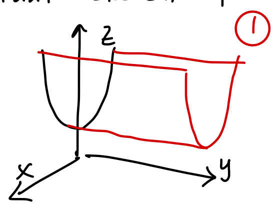
$$\boxed{V = \iint_R \left(f(x,y) - g(x,y)\right)\, dA}$$ if $f(x,y) \geq g(x,y)$ on R.

Ex 5: Find the volume of the solid bounded by the parabolic cylinder ① $z = 1 + x^2$ and the planes ③ $z = 5 - y$ and ② $y = 0$.

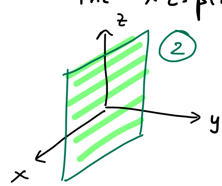
Sol: xz-trace of $z = 1 + x^2$:



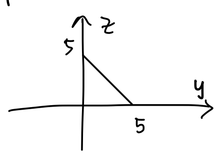
Actual sketch of $z = 1 + x^2$:



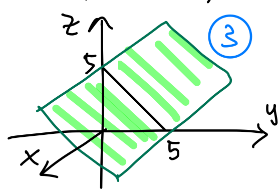
plane $y = 0$ is the xz-plane:

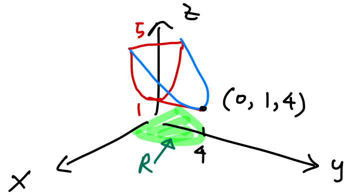


yz-trace of plane $z = 5 - y$:



Actual sketch of $z = 5 - y$:





Solid bounded by these 3 surfaces:

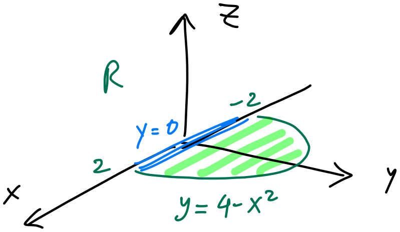
* upper surface is $\overbrace{z = 5 - y}^{③}$: Let $f(x,y) = 5 - y$
* lower surface is $\underbrace{z = 1 + x^2}_{①}$: Let $g(x,y) = 1 + x^2$
* Surfaces ① and ③ intersect along a curve C:

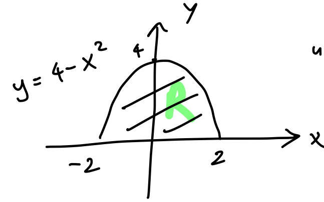
Set $5 - y = 1 + x^2$ and solve: $\boxed{4 - x^2 = y}$ ← this parabola is the projection of C onto the xy-plane

* The "back wall" of the solid is the plane $y = 0$. The projection of $y = 0$ onto the xy-plane is the x-axis.
* On the xy-plane, line $y = 0$ and parabola $y = 4 - x^2$ intersect when $0 = 4 - x^2 \iff x = 2, -2$.

(con't Ex 5)

* So the region of integration is $R = \{ (x,y) : 0 \le y \le 4-x^2,\ -2 \le x \le 2 \}$





upper bound $y = 4-x^2$

lower bound $y = 0$

left and right bounds are $-2 \le x \le 2$

* Volume of solid is $V = \iint_R \begin{pmatrix} \text{upper} \\ \text{surface} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{surface} \end{pmatrix} dA$

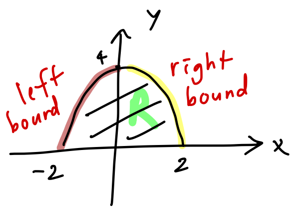
the plane (3) — the parabolic cylinder (1)

$$= \int_{-2}^{2} \underbrace{\boxed{\int_0^{4-x^2} (5-y) - (1+x^2)\, dy}}_{A(x)\ \text{(inner)}}\, dx$$

Do dy first because $0 \le y \le 4-x^2$ — non-constant bound

and $-2 \le x \le 2$ constant bounds

* Alternatively, we can change the order of integration:



$y = 4-x^2$

$x^2 = 4-y$

$x = -\sqrt{4-y}$, $x = \sqrt{4-y}$

(left bound) (right bound)

lower and upper bounds are $0 \le y \le 4$

$$V = \int_0^4 \underbrace{\boxed{\int_{-\sqrt{4-y}}^{\sqrt{4-y}} (5-y) - (1+x^2)\, dx}}_{A(y)\ \text{(inner)}}\, dy$$

Do dx first because $-\sqrt{4-y} \le x \le \sqrt{4-y}$ has non-constant bounds

(Con't Ex 5)

Let's do the first option (because integrating polynomials are easier):

$$A(x) = \int_0^{4-x^2} 5 - y - 1 - x^2 \, dy = \int_0^{4-x^2} [(4-x^2) - y] \, dy = (4-x^2)y - \frac{y^2}{2} \Big|_{y=0}^{y=4-x^2}$$

$$= \left((4-x^2)(4-x^2) - \frac{(4-x^2)^2}{2} \right) - (0)$$

$$= 16 - 8x^2 + x^4 - \frac{1}{2}(16 - 8x^2 + x^4)$$

$$= \frac{1}{2}[16 - 8x^2 + x^4]$$

$\boxed{16 - 8x^2 + x^4 \text{ is an even function}}$

$$V = \int_{-2}^{2} A(x) \, dx = \int_0^2 2A(x) \, dx = \int_0^2 16 - 8x^2 + x^4 \, dx = 16x - 8\frac{x^3}{3} + \frac{x^5}{5} \Big|_{x=0}^{x=2}$$

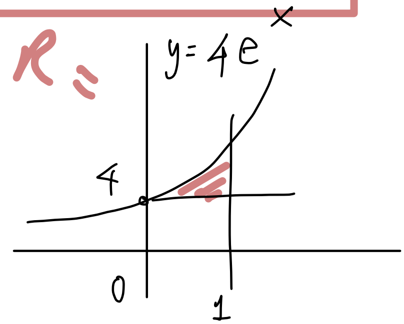
$$= \boxed{32 - \frac{64}{3} + \frac{64}{5}} = \frac{256}{15}$$

or do $\int_{-2}^{2} \frac{1}{2}[16 - 8x^2 + x^4] \, dx$, get the same answer.

— end of Ex 5 —

Def If R is a region in the xy-plane, the area of R is $\iint_R dA$

Ex: Area of $R =$



is $\displaystyle\int_0^1 \int_4^{4e^x} 1 \, dy \, dx = \int_0^1 \left(y \Big|_4^{4e^x} \right) dx$

$$= \int_0^1 \left(4e^x - 4\right) dx = 4e^x - 4x \Big|_{x=0}^{x=1}$$

$$= 4e - 4 - (4e^0 - 0) = 4e - 4 - 4 = \boxed{4e - 8}$$

Reading HW Sec 16.3 Double integrals in polar coordinates

Review examples of polar coordinates Sec 12.2

* Example 1 and 2 Convert between polar & Cartesian

* Example 4 Graph $r = 1 + \sin\theta$