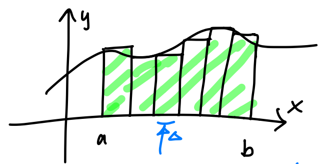
Sec 16.2 Reading HW:
Double integrals over general (non $\square$) regions: Read Example 1 & 2

Sec 16.1 Double integrals over rectangular regions

Idea: In calc 1, area under the curve $y=f(x)$ for $a \le x \le b$

is taking limit as $\Delta \to 0$ of the sum of areas of (vertical) rectangles.



each rectangle has width Δ.
Height of rectangle depends on $f(x)$

Area is the definite integral

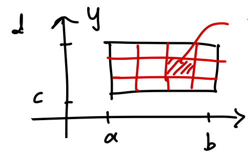
denoted $\int_a^b f(x)\,dx$

Now:

volume under the surface $z=f(x,y)$ over

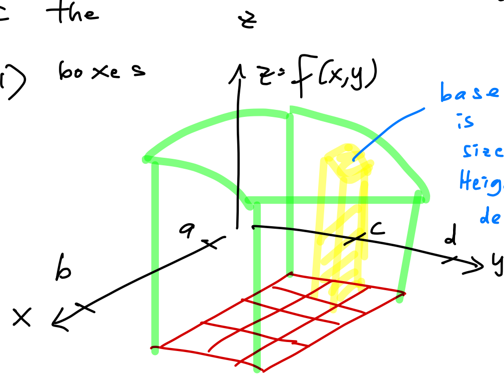
the rectangle $R=[a,b]\times[c,d]$

meaning $R=\{(x,y): a\le x\le b, c\le y\le d\}$



square of size Δ by Δ

is taking limit as $\Delta \to 0$ of the sum of volumes of (vertical) boxes



base of each box is square of size $\Delta\times\Delta$.
Height of box depends on $f(x,y)$

Volume is the double integral of $f(x,y)$ over R

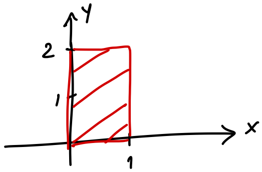
(where R is a rectangular region in the xy-plane)

denoted $\iint_R f(x,y)\,dA$.

Ex 1 & 2:

Let $f(x,y) := \overbrace{6-2x-y}$

Consider the solid region bounded by the plane $z = 6-2x-y$ over the rectangular region $R: \{(x,y) : 0 \le x \le 1, 0 \le y \le 2\}$ or $[0,1] \times [0,2]$ for short

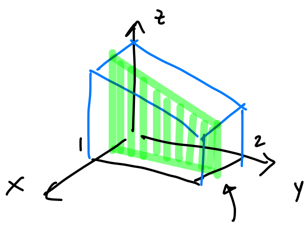


The volume of this solid is the double integral

$$V = \iint_R f(x,y)\, dA = \iint_R (6-2x-y)\, dA$$

How to compute V:

Keep x value constant, take a vertical slice of the solid parallel to the yz-plane.



This slice at point x has a cross-sectional area denoted $A(x)$.

$$A(x) = \int_0^2 (6-2x-y)\, dy$$

$$= 6y - 2xy - \frac{y^2}{2} \Big|_{y=0}^{y=2}$$

$$= (12 - 4x - 2) - (0)$$

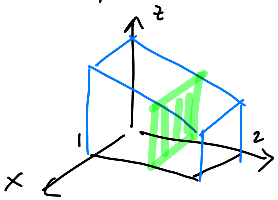
$$= 10 - 4x$$

To compute the volume of the solid, we use <u>iterated integral</u> (meaning repeated integral):

$$V = \int_0^1 A(x)\, dx = \int_0^1 \Big[\overbrace{\int_0^2 (6-2x-y)\, dy}^{A(x)} \Big] dx$$

$$= \int_0^1 10 - 4x\, dx = 10x - \frac{4x^2}{2} \Big|_0^1 = (10-2) - 0 = \boxed{8}$$

If we first slice the solid by keeping y constant,



we would get $A(y) = \int_0^1 (6 - 2x - y)\, dx$.

$= 5 - y$

Volume is $V = \int_0^2 A(y)\, dy = \int_0^2 \left[\overbrace{\int_0^1 (6 - 2x - y)\, dx}^{A(y)} \right] dy$

$= \int_0^2 5 - y\, dy = 5y - \frac{y^2}{2} \Big|_0^2 = 8$ (same answer)

Thm (Fubini)

(order of integration can be swapped)

The double integral of f over a rectangular region $R = [a, b] \times [c, d]$ may be evaluated by either of two iterated integrals:

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

Def: The <u>average value</u> of $f(x,y)$ over region R is

$$\frac{1}{\text{area of } R} \iint_R f(x,y)\, dA$$

(MML #8)

The average value of $f(x,y) = \cos x \sin y$ over the region $R = [0, \frac{\pi}{3}] \times [0, \frac{\pi}{4}]$ is

$$\frac{1}{\text{area of } R} \iint_R f(x,y)\, dA = \frac{1}{\underbrace{\left(\frac{\pi}{3} \cdot \frac{\pi}{4}\right)}_{\frac{\pi^2}{12}}} \underbrace{\int_0^{\frac{\pi}{4}} \overbrace{\int_0^{\frac{\pi}{3}} \cos x \sin y \, dx}^{\text{inner integral}} \, dy}_{\text{outer integral}}$$

(Here I choose my inner integral to be wrt x)

<u>inner</u>

$$\int_0^{\frac{\pi}{3}} \cos x \sin y \, dx = \sin y \int_0^{\frac{\pi}{3}} \cos x \, dx$$

$$= \sin y \sin x \Big|_{x=0}^{x=\frac{\pi}{3}}$$

$$= \sin y \left(\sin \frac{\pi}{3} - \sin 0\right) = \frac{\sqrt{3}}{2} \sin y$$

<u>outer</u>

$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{3}}{2} \sin y \, dy = -\frac{\sqrt{3}}{2} \cos y \Big|_{y=0}^{y=\frac{\pi}{4}}$$

$$= -\frac{\sqrt{3}}{2}\left(\cos \frac{\pi}{4} - \cos 0\right)$$

$$= -\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2} - 1\right) = \frac{\sqrt{3}}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$$

The average value is $\frac{12}{\pi^2} \frac{\sqrt{3}}{2}\left(1 - \frac{\sqrt{2}}{2}\right) = \boxed{\frac{6\sqrt{3}}{\pi^2}\left(1 - \frac{\sqrt{2}}{2}\right)}$