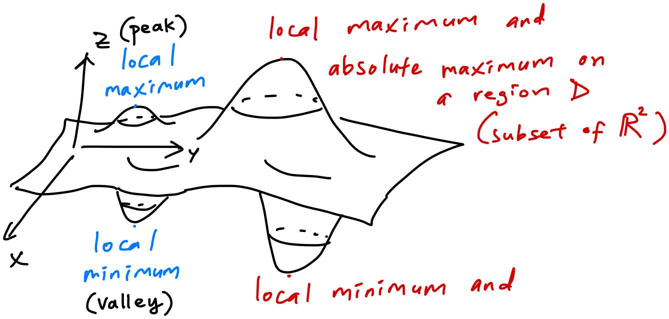
15.7 Maximum/minimum problems



Def:
- If $f(x,y) \le f(a,b)$ for all (x,y) in some open disk centered at (a,b), then $f(a,b)$ is a <u>local maximum value of f</u>.
- If $f(x,y) \ge f(a,b)$ for all (x,y) in some open disk centered at (a,b), then $f(a,b)$ is a <u>local minimum value of f</u>. These are also called <u>cal extreme values</u>.

(Think: If you're standing at a local maximum, you cannot walk uphill.
" " minimum, " " downhill.)

Thm If f has a local max or min value at (a,b) and the partial derivatives f_x and f_y exist at (a,b), then $f_x(a,b) = f_y(a,b) = 0$.

Note The other direction is not true!

Def Let (a,b) be an interior point in the domain of f.

Then (a,b) is a critical point if either

1) $f_x(a,b) = f_y(a,b) = 0$, OR

2) f_x doesn't exist at (a,b), OR

3) f_y doesn't exist at (a,b).

Ex 1 : Find the critical points of $f(x,y) = xy(x-2)(y+3)$

Sol: $\frac{\partial}{\partial x}\left(y(x^2-2x)(y+3)\right) = y(y+3)(2x-2) = 2y(y+3)(x-1)$

$\frac{\partial}{\partial y}\left(x(x-2)(y^2+3y)\right) = x(x-2)(2y+3)$

Both partial derivatives exist for all (a,b) in $\mathbb{R}^2$.

Set both $f_x(x,y) \overset{(eq\,1)}{=} 0$ and $f_y(x,y) \overset{(eq\,2)}{=} 0$.

$f_x(x,y) \overset{(eq\,1)}{=} 0 \iff$ (i) $y=0$, (ii) $y=-3$, (iii) $x=1$

(i) Sub $y=0$ into $x(x-2)(2y+3) \overset{(eq\,2)}{=} 0 \Rightarrow x(x-2)3=0$

$\Rightarrow x=0,\ x=2$

So $\boxed{(0,0), (2,0)}$ are critical points

(ii) Sub $y=-3$ into $x(x-2)(2y+3) \overset{(eq\,2)}{=} 0 \Rightarrow x(x-2)(2(-3)+3)=0$

$\Rightarrow x=0,\ x=2$

So $\boxed{(0,-3), (2,-3)}$ are critical points.

(iii) Sub $x=1$ into $x(x-2)(2y+3) \overset{(eq\,2)}{=} 0 \Rightarrow 1(1-2)(2y+3)=0$

$\Rightarrow 2y=-3 \Rightarrow y=-\frac{3}{2}$

So $\boxed{(1,-\frac{3}{2})}$ is a critical point.

There are five critical points total.

Students try

Def: Suppose $f_x(a,b) = f_y(a,b)$.

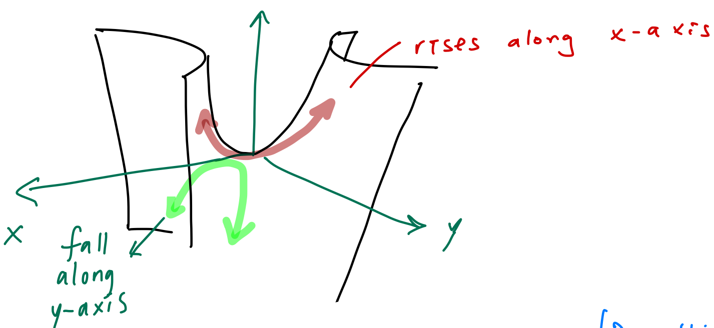
Then f has a <u>saddle point</u> at (a,b) if, in every open disk centered at (a,b),

there are points (x,y) for which $f(x,y) > f(a,b)$

and points (x,y) for which $f(x,y) < f(a,b)$.

Think: If you're standing at a saddle point, it's possible to walk uphill in some directions and downhill in other directions.

Ex: $z = x^2 - y^2$ has a saddle point at the origin.



[Recall: $f_{xy} = f_{yx}$ whenever f_x, f_y are continuous in an open set (Sec 15.3)]

Def

$$D(x,y) := \underbrace{\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}}_{\text{Called Hessian matrix}} = f_{xx} f_{yy} - (f_{xy})^2$$ is called the <u>discriminant</u> of f.

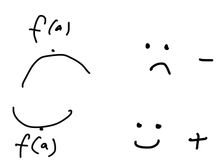
2x2 determinant

Recall 2nd Derivative Test from Calc I:

2nd Derivative Test:

If $f'(a)=0$ & $f''(a)<0$, then f has a local max

If $f'(a)=0$ & $f''(a)>0$, then f has a local min



If $f''(a)=0$, the test is inconclusive

Calc III Second Derivative Test

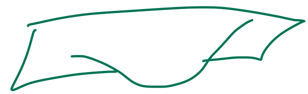
* Suppose f_x, f_y are continuous throughout an open disk centered at the point (a,b).

* Suppose $f_x(a,b) = f_y(a,b) = 0$.

1) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum value at (a,b).

discriminant

2) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, — " — minimum



3) If $D(a,b) < 0$, then f has a saddle point at (a,b).

4) If $D(a,b) = 0$, then the test is inconclusive
(ie. (a,b) may correspond to case 1 or 2 or 3 or not)

Note: $D(a,b) > 0$ means the surface has the same general behavior in all directions near (a,b)

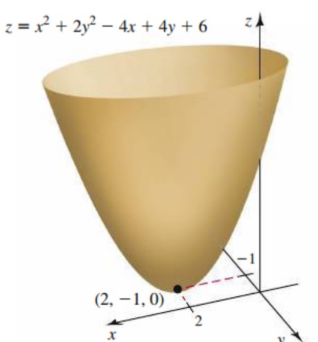
EXAMPLE 2 **Analyzing critical points** Use the Second Derivative Test to classify the critical points of $f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$.

SOLUTION We begin with the following derivative calculations:

$$f_x = 2x - 4, \quad f_y = 4y + 4,$$
$$f_{xx} = 2, \quad f_{xy} = f_{yx} = 0, \quad \text{and} \quad f_{yy} = 4.$$

Setting both f_x and f_y equal to zero yields the single critical point $(2, -1)$. The value of the discriminant at the critical point is $D(2, -1) = f_{xx} f_{yy} - (f_{xy})^2 = 8 > 0$. Furthermore, $f_{xx}(2, -1) = 2 > 0$. By the Second Derivative Test, f has a local minimum at $(2, -1)$; the value of the function at that point is $f(2, -1) = 0$ (**Figure 15.69**).

Related Exercise 24



Local minimum at $(2, -1)$ where $f_x = f_y = 0$

(Additional Example)

Ex 3   Same $f(x,y) = xy(x-2)(y+3)$ as Ex 1.

We found five critical points $(0,0)$, $(2,0)$, $(1,-\frac{3}{2})$, $(0,-3)$, $(2,-3)$.

EXAMPLE 3 **Analyzing critical points** Use the Second Derivative Test to classify the critical points of $f(x, y) = xy(x - 2)(y + 3)$.

SOLUTION In Example 1, we determined that the critical points of f are $(0, 0)$, $(2, 0)$, $\left(1, -\frac{3}{2}\right)$, $(0, -3)$, and $(2, -3)$. The derivatives needed to evaluate the discriminant are

$$f_x = 2y(x - 1)(y + 3), \qquad f_y = x(x - 2)(2y + 3),$$
$$f_{xx} = 2y(y + 3), \qquad f_{xy} = 2(2y + 3)(x - 1), \quad \text{and} \quad f_{yy} = 2x(x - 2).$$

The values of the discriminant at the critical points and the conclusions of the Second Derivative Test are shown in Table 15.4.

Table 15.4

(x, y)	$D(x, y)$	f_{xx}	Conclusion
$(0, 0)$	-36	0	Saddle point
$(2, 0)$	-36	0	Saddle point
$\left(1, -\frac{3}{2}\right)$	9	$-\frac{9}{2}$	Local maximum
$(0, -3)$	-36	0	Saddle point
$(2, -3)$	-36	0	Saddle point

Absolute Maximum and Minimum Values

As in the one-variable case, we are often interested in knowing where a function of two or more variables attains its extreme values over its domain (or a subset of its domain).

DEFINITION **Absolute Maximum/Minimum Values**

Let f be defined on a set R in $\mathbb{R}^2$ containing the point (a, b). If $f(a, b) \geq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute maximum value** of f on R. If $f(a, b) \leq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute minimum value** of f on R.

Fact: Absolute max & min on a closed bounded set R occur as local max/min at interior points

OR

they occur on the boundary of R

Ex 5: (MML #8)

A company makes rectangular boxes s.t the sum of the length, width, & height of the box is equal to 96 inches. Find the dimensions of the box that has the largest volume.

Sol: Let x, y, z be the length, width, height of the box.

must be nonnegative

The volume of the box is $V = xyz$.

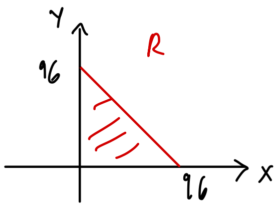
This restriction $x+y+z = 96$ can be used to eliminate one of the 3 variables : $z = 96 - x - y$, so the volume function becomes a two-variable function

$$V(x,y) = xy(96 - x - y)$$

Since length, width, height are nonnegative, we have constraints

$$x \geq 0, \quad y \geq 0, \quad \underbrace{96 - x - y \geq 0}_{96 \geq x+y}$$

So the domain of of $V(x,y)$ is $R := \{ (x,y) : 0 \leq x \leq 96, \ 0 \leq y \leq 96, \ x + y \leq 96 \}$



The boundary of R is :

the line segment $x = 0$ where $0 \leq y \leq 96$,

— " — $y = 0$ where $0 \leq x \leq 96$,

— " — $x + y = 96$ where $0 \leq x \leq 96$

Check: On the boundary of R, $V(x,y) = 0$. So a maximum of $V(x,y)$, if any, must occur at an interior point.

To determine the behavior of V at interior points of R, find critical points (when $V_x = 0 = V_y$, or V_x or V_y doesn't exist):

$$V = 96xy - x^2y - y^2x$$

$$V_x = 96y - 2xy - y^2 = y(96 - 2x - y)$$

$$V_y = 96x - x^2 - 2yx = x(96 - x - 2y)$$

both always exist

Set $V_x = 0$: $y(96 - 2x - y) = 0$

$y = 0$ or $96 - 2x - y = 0$

$y = 96 - 2x$

Sub $y=0$ into $V_y = 0$: $x(96 - x) = 0$

$x = 0$ or $x = 96$

$$\boxed{(0,0) \qquad (96,0)}$$

Sub $y = 96 - 2x$ into $V_y = 0$: $x(96 - x - 2(96 - 2x)) = 0$

$x(96 - x - 2(96) + 4x) = 0$

$x(-96 + 3x) = 0$

$x = 0$ or $-96 + 3x = 0$

$3x = 96$

$x = 32$

$(0, 96 - 2(0))$ $\quad$ $(32, 96 - 2(32))$

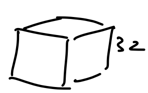
$\boxed{(0,96)}$ $\quad$ $\boxed{(32,32)}$

- There are 4 critical points:

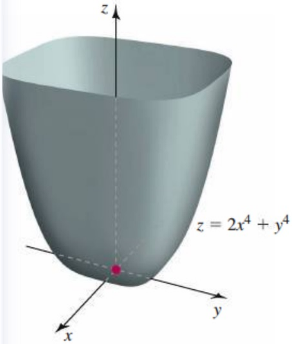
* $(0,0)$, $(96,0)$, $(0,96)$ give volume 0.

* At $(32,32)$, volume is $V(32,32) = (32)(32)(96 - 32 - 32) = \boxed{(32)^3}$

- So box dimensions w/ max volume are $32, 32, 32$.

Cube 

(Additional Example)

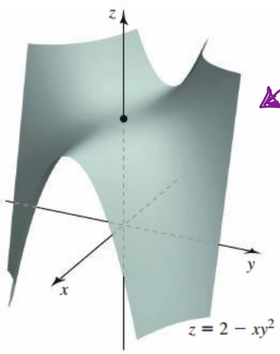


Local minimum at (0, 0), but the Second Derivative Test is inconclusive.

Figure 15.71

➤ The same "flat" behavior occurs with functions of one variable, such as $f(x) = x^4$. Although f has a local minimum at $x = 0$, the Second Derivative Test is inconclusive.

➤ It is not surprising that the Second Derivative Test is inconclusive in Example 4b. The function has a line of local maxima at $(a, 0)$ for $a > 0$, a line of local minima at $(a, 0)$ for $a < 0$, and a saddle point at $(0, 0)$.



Second derivative test fails to detect saddle point at (0, 0).

EXAMPLE 4 **Inconclusive tests** Apply the Second Derivative Test to the following functions and interpret the results.

a. $f(x, y) = 2x^4 + y^4$ **b.** $f(x, y) = 2 - xy^2$

SOLUTION

a. The critical points of f satisfy the conditions

$$f_x = 8x^3 = 0 \quad \text{and} \quad f_y = 4y^3 = 0,$$

so the sole critical point is $(0, 0)$. The second partial derivatives evaluated at $(0, 0)$ are

$$f_{xx}(0, 0) = f_{xy}(0, 0) = f_{yy}(0, 0) = 0.$$

We see that $D(0, 0) = 0$, and the Second Derivative Test is inconclusive. While the bowl-shaped surface (Figure 15.71) described by f has a local minimum at $(0, 0)$, the surface also has a broad flat bottom, which makes the local minimum "invisible" to the Second Derivative Test.

b. The critical points of this function satisfy

$$f_x(x, y) = -y^2 = 0 \quad \text{and} \quad f_y(x, y) = -2xy = 0.$$

The solutions of these equations have the form $(a, 0)$, where a is a real number. It is easy to check that the second partial derivatives evaluated at $(a, 0)$ are

$$f_{xx}(a, 0) = f_{xy}(a, 0) = 0 \quad \text{and} \quad f_{yy}(a, 0) = -2a.$$

Therefore, the discriminant is $D(a, 0) = 0$, and the Second Derivative Test is inconclusive. Figure 15.72 shows that f has a flat ridge above the x-axis that the Second Derivative Test is unable to classify.