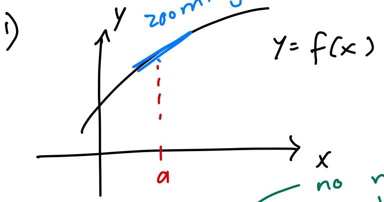
15.6 Tangent planes & linear approximations

I. Tangent planes II. linear approximations

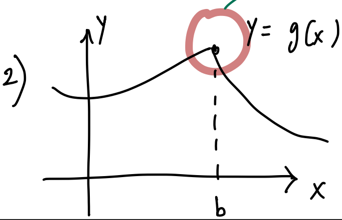
Part I

In Calc 1: zooming in on $(x, f(x))$, the curve looks like the tangent line at $(a, f(a))$



no matter how much we zoom in, the curve will not look like a line

No tangent line at $(b, g(b))$



Now

Consider a surface $\boxed{z = f(x,y)}$ (explicit form) or $\boxed{f(x,y) - z = 0}$

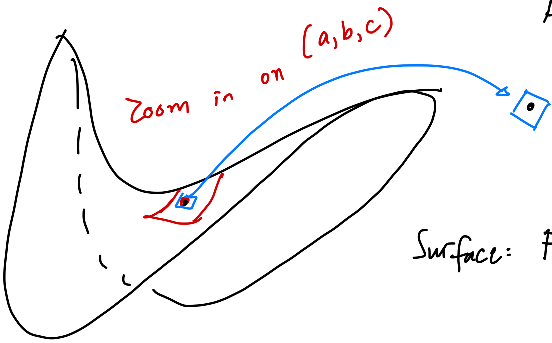
Equivalently, if $F(x,y,z) \overset{def}{:=} f(x,y) - z$, then

$\boxed{F(x,y,z) = 0}$ (implicit form) describes the same surface.

Ex: $\boxed{z = \underbrace{xy + x - y}_{f(x)}}$ is an explicit form.

An implicit form is $\boxed{\underbrace{xy + x - y - z}_{F(x,y,z)} = 0}$

Idea

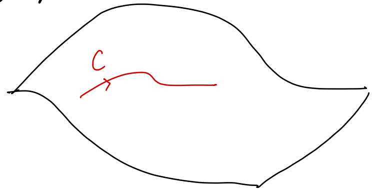


As we zoom in more and more on the point (a,b,c), the surface looks more and more like a plane.

Surface: $F(x,y,z) = 0$.

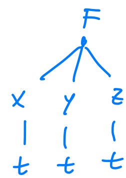
What is a tangent plane?

- Consider a curve $C : \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ that lies on the surface $F(x,y,z) = 0$.



- Because all points on C lie on the surface $F(x,y,z)=0$, we have $F(x(t), y(t), z(t)) = 0$.

- Differentiate w/ respect to t:



$$\frac{d}{dt} F(x(t), y(t), z(t)) = \frac{d}{dt} 0$$

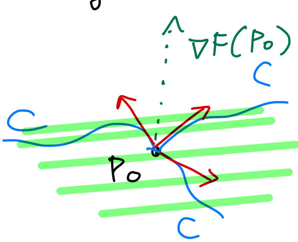
$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

$$\underbrace{\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle}_{\text{the gradient } \nabla F(x,y,z)} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle}_{\vec{r}\,'(t)} = 0$$

This means: **At any point on the curve C, the tangent vector $\vec{r}\,'(t)$ is orthogonal to $\nabla F(x,y,z)$**

Let $P_0(a,b,c)$ be a point on the surface where the gradient $\nabla F(a,b,c) \neq \langle 0,0,0 \rangle$.

For all curves C on the surface passing through $P_0(a,b,c)$, any vector tangent to C is orthogonal to $\nabla F(a,b,c)$.



All these tangent vectors lie in the same plane, the tangent plane at P_0.

This plane has a normal vector $\nabla F(a,b,c)$ and this plane contains a point $P_0(a,b,c)$.

$\underline{\text{Def}}$ The $\underline{\text{tangent plane}}$ of the surface $\underbrace{F(x,y,z)=0}_{\text{(Implicit form)}}$

at $P_0\,(a,b,c)$ is the plane ...

(1) containing the point P_0

(2) orthogonal to the gradient $\nabla F(a,b,c)$

So an equation of the tangent plane of the surface $F(x,y,z)=0$ at point P_0 is given by

$$\boxed{\nabla F(P_0)\cdot\overrightarrow{P_0P}=0}$$ — See Sec 13.5 (lines & planes in space)

that is, $\langle F_x(P_0), F_y(P_0), F_z(P_0)\rangle\cdot\langle x-a, y-b, z-c\rangle=0$, or,

equivalently, $$\boxed{F_x(P_0)\,(x-a)+F_y(P_0)\,(y-b)+F_z(P_0)\,(z-c)=0}$$

Ex 1: Consider the ellipsoid $\dfrac{x^2}{9}+\dfrac{y^2}{25}+z^2=1$ (See Sec 13.6)

a) Find the tangent plane to the surface at $\underbrace{(0,4,\tfrac{3}{5})}_{\text{our } P_0(a,b,c)}$.

Sol: Rewrite $\underbrace{\dfrac{x^2}{9}+\dfrac{y^2}{25}+z^2-1}=0$

Let $F(x,y,z):=\dfrac{x^2}{9}+\dfrac{y^2}{25}+z^2-1$

$\nabla F(x,y,z)=\langle F_x, F_y, F_z\rangle=\langle \frac{2}{9}x, \frac{2}{25}y, 2z\rangle$

$\nabla F(0,4,\frac{3}{5})=\langle 0, \frac{2}{25}(4), 2(\frac{3}{5})\rangle=\langle 0, \frac{8}{25}, \frac{6}{5}\rangle$ is a normal vector to the tangent plane at P_0

Eq of the tangent plane is

$$\boxed{0\,(x-0)+\frac{8}{25}(y-4)+\frac{6}{5}\left(z-\frac{3}{5}\right)=0}$$

cont ↓

(con't) b) At what points (a,b,c) on the surface is the tangent plane horizontal?

Sol: A horizontal plane is of the form $z = c$

so it has (vertical) normal vector of the form $\langle 0, 0, z_0 \rangle$. $\hookrightarrow z_0 \neq 0$

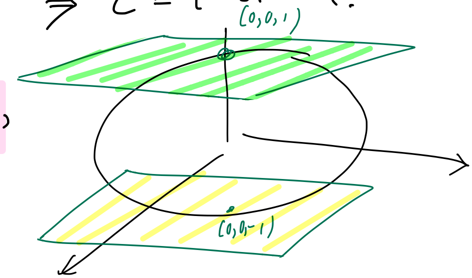
This normal vector is parallel to $\nabla F(a,b,c)$.

In part (a), we computed $\nabla F(x,y,z) = \left\langle \frac{2}{9}x, \frac{2}{25}y, 2z \right\rangle$

$$\text{So} \quad \frac{2}{9}x = 0, \quad \frac{2}{25}y = 0, \quad 2z = c$$

$$\text{So} \quad x = 0, \quad y = 0$$

$$F(0,0,c) = 0 \Rightarrow \frac{0^2}{9} + \frac{0^2}{25} + c^2 - 1 = 0 \Rightarrow c = 1 \text{ or } -1.$$



Answer: At $\boxed{\text{points } (0,0,1) \text{ and } (0,0,-1)}$, the tangent planes are horizontal.

Fact The tangent plane of the surface $\underbrace{z = f(x,y)}_{\text{(Explicit form)}}$

at $P_0\,(a, b, \overset{c}{\overbrace{f(a,b)}})$ is the plane ...

(1) containing the point P_0

(2) orthogonal to $\langle f_x(a,b), f_y(a,b), -1 \rangle$ $\quad \frac{\partial}{\partial z}[f(x,y) - z]$

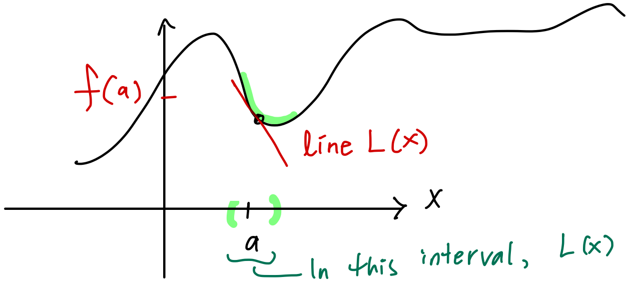
So an equation of the tangent plane of the surface $z = f(x,y)$ at point P_0 is given by

$$f_x(P_0)(x-a) + f_y(P_0)(y-b) - 1\,(z-c) = 0 \qquad c = f(a,b)$$

Equiv, $\boxed{z = f_x(P_0)(x-a) + f_y(P_0)(y-b) + f(a,b)}$ (*)

Part II. Linear approximation.

Calc 1: $y = f(x)$



In this interval, $L(x)$ and $f(x)$ are close.

The line $L(x)$ tangent to the curve $y = f(x)$ at point $(a, f(a))$

is $\underbrace{f'(a)}_{\text{(slope)}} = \dfrac{\overbrace{L(x)}^{y} - \overbrace{f(a)}^{y_0}}{x - \underset{x_0}{a}}$ or $\boxed{L(x) = f'(a)(x-a) + f(a)}$

This tangent line L gives a (linear) approximation to f at $x=a$.

At points near $x=a$, we have $f(x) \underset{\text{close}}{\approx} L(x)$

Now for $z = f(x,y)$.

The plane $L(x,y)$ tangent to the surface $z = f(x,y)$ at point $(a, b, f(a,b))$

is $\boxed{L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)}$

(See (*) above. We replace z with its approximation $L(x,y)$)

This L gives a (linear) approximation to f at $(x,y) = (a,b)$.

At points near $(x,y) = (a,b)$, we have $f(x,y) \underset{\text{close}}{\approx} L(x,y)$.

Ex 3: Let $f(x,y) = \dfrac{5}{x^2+y^2} = 5(x^2+y^2)^{-1}$

a) Find the linear approximation for f at $(-1, 2, 1)$. (a, b, $f(a,b)$)

Sol:

$$f_x = 5(-1)(x^2+y^2)^{-2}\, 2x = \boxed{-10x \frac{1}{(x^2+y^2)^2}} \qquad f_x(-1,2) = -10(-1)\frac{1}{(1^2+2^2)^2} = \frac{10}{25} = \frac{2}{5}$$

$$f_y = \boxed{-10y \frac{1}{(x^2+y^2)^2}} \qquad f_y(-1,2) = -10(2)\frac{1}{25} = \frac{-20}{25} = -\frac{4}{5}$$

$$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

$$L(x,y) = \frac{2}{5}(x-(-1)) + -\frac{4}{5}(y-2) + 1$$

$$= \frac{2}{5}x + \frac{2}{5} - \frac{4}{5}y + \frac{8}{5} + 1$$

$$= \boxed{\frac{2}{5}x - \frac{4}{5}y + 3}$$

b) Use the linear approximation (part (a)) to estimate the value of $f(-1.05, 2.1)$.

Sol: $L(-1.05, 2.1) = \frac{2}{5}(-1.05) - \frac{4}{5}(2.1) + 3 = \frac{9}{10} = 0.90$

Note: Actual value is about 0.907, & the relative error is $\frac{0.007}{0.907} = \frac{7}{907}$, less than 0.8 %

Def ("Approximate change formula") MML # 2

The change in $z = f(x,y)$ as (x,y) changes to $(x+dx, y+dy)$ is denoted by Δz.

This change is approximated by the differential dz.

$$\Delta z \approx \boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy}$$

Reading HW: Read Sec 15.7 (max/min problems) Examples 1 & 2 about critical points