15.5 Directional derivatives and the gradient

Goals: You're standing on a surface.

① You walk in a direction not parallel to the x-axis or y-axis.
What is the rate of change of in this direction ?

② You release a ball and let it roll.
In which direction will it roll ?

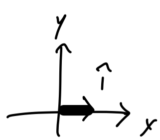
③ If you are hiking up the surface (say, it's a mountain), in what direction should you walk after each step if you want to follow the steepest path?

Thm / Def

Let f be differentiable at (a,b),
$\vec{u} = \langle u_1, u_2 \rangle$ a unit vector in the xy-plane.
The directional derivative of f at (a,b) in the direction of $\vec{u}$
is
$$D_{\vec{u}} f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle$$
$$= u_1 f_x(a,b) + u_2 f_y(a,b).$$

Note:

If $\vec{u} = \hat{i} = \langle 1, 0 \rangle$



$D_{\langle 1,0 \rangle} f(a,b) = \underbrace{f_x(a,b)}$,
(the partial derivative wrt x)

If $\vec{u} = \hat{j} = \langle 0, 1 \rangle$, $D_{\langle 0,1 \rangle} f(a,b) = f_y(a,b)$

(Note: The vector $\langle f_x(a,b), f_y(a,b)\rangle$ that appears in the dot product for directional derivatives is called the <u>gradient</u> of f)

Def The gradient of a differentiable function f at (x,y) is the vector-valued function

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = f_x(x,y)\hat{\imath} + f_y(x,y)\hat{\jmath}$$

Note: We can now write $D_{\vec{u}} f(a,b) = \nabla f(a,b)\cdot\vec{u}$.

Ex 1: Let $f(x,y) = \dfrac{x^2}{4} + \dfrac{y^2}{2} + 2$.

The gradient of f is $\nabla f = \langle f_x, f_y\rangle = \langle \frac{2}{4}x, y\rangle$.

Let P_0 be the point $(3,2)$ on the xy-plane.

Let $\vec{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$, a unit vector.

$$D_{\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle} f(3,2) = \left\langle f_x(3,2), f_y(3,2)\right\rangle \cdot \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$$

$$= \left\langle \frac{2x}{4}\Big|_{(3,2)}, y\Big|_{(3,2)}\right\rangle \cdot \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$$

$$= \left\langle \frac{2}{4}(3), 2\right\rangle \cdot \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$$

$$= \frac{1}{2}(3)\frac{\sqrt{2}}{2} + 2\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{4} + \sqrt{2} = \boxed{\frac{7\sqrt{2}}{4}}$$

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Ex 1: Let $f(x,y) = \dfrac{x^2}{4} + \dfrac{y^2}{2} + 2$.

(Review)

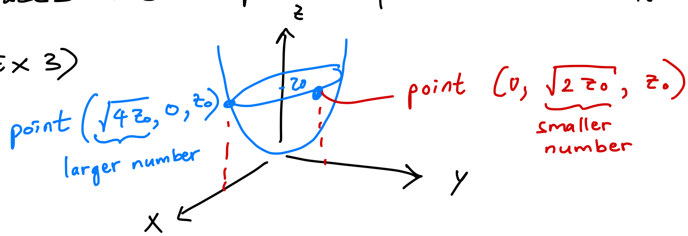
Then its graph is $z = \dfrac{x^2}{4} + \dfrac{y^2}{2} + 2$ is the paraboloid

$z = \dfrac{x^2}{4} + \dfrac{y^2}{2}$ shifted up by $+2$

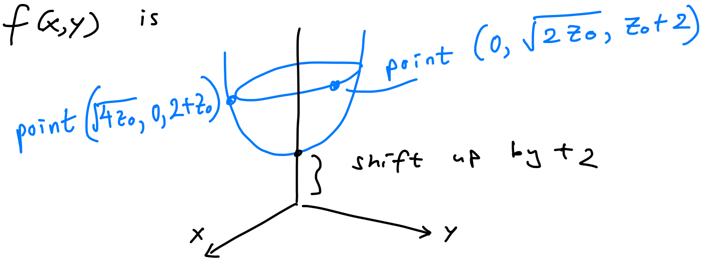
Recall (Sec 13.6) that a horizontal trace of a surface is the set of points of the surface which intersects a horizontal plane ($z = z_0$).

The horizontal traces are ellipses $\dfrac{x^2}{4} + \dfrac{y^2}{2} = z_0$ or $\dfrac{x^2}{4z_0} + \dfrac{y^2}{2z_0} = 1$

(Review Sec 13.6 Ex 3)



So our surface $z = f(x,y)$ is



Let P_0 be the point $(3,2)$ on the xy-plane.

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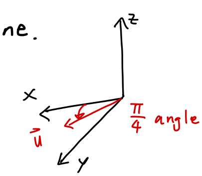
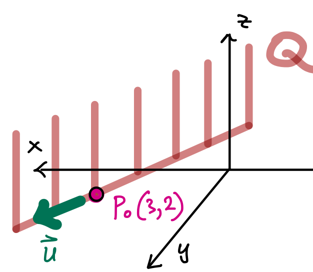


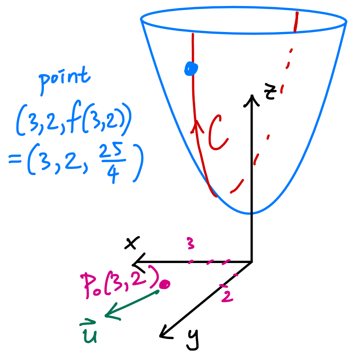
Let Q be the vertical plane containing $P_0(3,2)$ and $\vec{u}$



Now, think about slicing the surface with Q:

Let C be the curve along which the surface intersects Q.

$$f(x,y) = \frac{x^2}{4} + \frac{y^2}{2} + 2$$



The slope of the line tangent to C at point $(3, 2, \frac{25}{4})$ is the directional derivative of f in the direction of $\vec{u}$, $D_{\vec{u}}\, f(3,2)$.

We expect this number to be positive because the curve is pointing up as we walk along the curve (in dir of $\vec{u}$) from point $(3, 2, \frac{25}{4})$.

We compute $D_{\vec{u}}\, f(3,2)$ using the dot product formula:

We computed this earlier

$$D_{\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle} f(3,2) = \left\langle f_x(3,2),\ f_y(3,2) \right\rangle \cdot \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

$$= \left\langle \frac{2x}{4}\Big|_{(3,2)},\ y\Big|_{(3,2)} \right\rangle \cdot \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

$$= \left\langle \frac{2}{4}(3),\ 2 \right\rangle \cdot \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

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