Reading HW Sec 15.6

(Tangent planes & linear approximation)

Read Example 1 or 2

15.5 Directional derivatives and the gradient

Goals: You're standing on a surface.

① You walk in a direction not parallel to the x-axis or y-axis.

What is the rate of change of in this direction?

② You release a ball and let it roll.

In which direction will it roll?

③ If you are hiking up the surface (say, it's a mountain), in what direction should you walk after each step if you want to follow the steepest path?

Thm / Def

Let f be differentiable at (a,b),

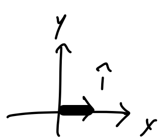
$\vec{u} = \langle u_1, u_2 \rangle$ a unit vector in the xy-plane.

The directional derivative of f at (a,b) in the direction of $\vec{u}$

is
$$D_{\vec{u}} f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle$$
$$= u_1 f_x(a,b) + u_2 f_y(a,b).$$

Note:

If $\vec{u} = \hat{i} = \langle 1, 0 \rangle$



$D_{\langle 1,0 \rangle} f(a,b) = \underbrace{f_x(a,b)}$,

(the partial derivative wrt x)

If $\vec{u} = \hat{j} = \langle 0, 1 \rangle$, $D_{\langle 0,1 \rangle} f(a,b) = f_y(a,b)$

(Note: The vector $\langle f_x(a,b), f_y(a,b)\rangle$ that appears in the dot product for directional derivatives is called the <u>gradient</u> of f)

<u>Def</u> The gradient of a differentiable function f at (x,y) is the vector-valued function

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = f_x(x,y)\hat{\imath} + f_y(x,y)\hat{\jmath}$$

<u>Note:</u> We can now write $D_{\vec{u}} f(a,b) = \nabla f(a,b)\cdot\vec{u}$.

Ex 1: Let $f(x,y) = \dfrac{x^2}{4} + \dfrac{y^2}{2} + 2$.

The gradient of f is $\nabla f = \langle f_x, f_y\rangle = \langle \frac{2}{4}x, y\rangle$.

Let P_0 be the point $(3,2)$ on the xy-plane.

Let $\vec{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$, a unit vector.

$$D_{\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle} f(3,2) = \left\langle f_x(3,2),\ f_y(3,2)\right\rangle \cdot \langle \tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\rangle$$

$$= \left\langle \frac{2x}{4}\Big|_{(3,2)},\ y\Big|_{(3,2)}\right\rangle \cdot \langle \tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\rangle$$

$$= \langle \tfrac{2}{4}(3),\ 2\rangle \cdot \langle \tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\rangle$$

$$= \frac{1}{2}(3)\frac{\sqrt{2}}{2} + 2\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{4} + \sqrt{2} = \boxed{\frac{7\sqrt{2}}{4}}$$

~ ended here Week 6 Friday —

Ex 1: Let $f(x,y) = \dfrac{x^2}{4} + \dfrac{y^2}{2} + 2$.

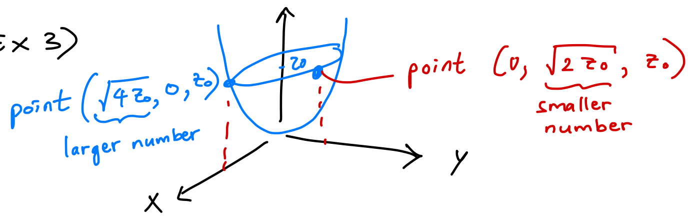
Then its graph is $z = \dfrac{x^2}{4} + \dfrac{y^2}{2} + 2$ is the paraboloid

(Review) $\quad z = \dfrac{x^2}{4} + \dfrac{y^2}{2}$ shifted up by $+2$

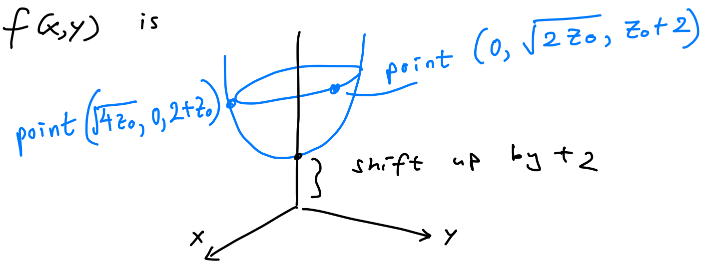
Recall (Sec 13.6) that a horizontal trace of a surface is the set of points of the surface which intersects a horizontal plane $(z = z_0)$.

The horizontal traces are ellipses $\dfrac{x^2}{4} + \dfrac{y^2}{2} = z_0$ or $\dfrac{x^2}{4z_0} + \dfrac{y^2}{2z_0} = 1$

(Review Sec 13.6 Ex 3)

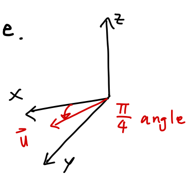


So our surface $z = f(x,y)$ is

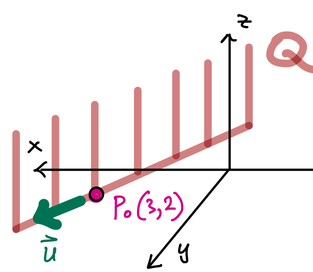


Let P_0 be the point $(3,2)$ on the xy-plane.

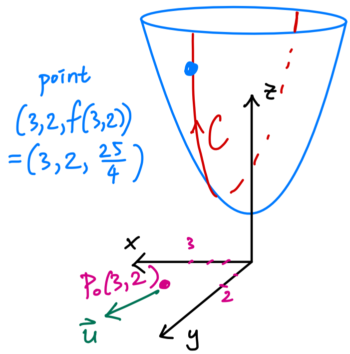
Let $\vec{u} = \left\langle \dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2} \right\rangle$, a unit vector



Let Q be the vertical plane containing $P_0(3,2)$ and $\vec{u}$



Now, think about slicing the surface with Q:
Let C be the curve along which the surface intersects Q.



$$f(x,y) = \frac{x^2}{4} + \frac{y^2}{2} + 2$$

The slope of the line tangent to C at point $(3,2,\frac{25}{4})$ is the directional derivative of f in the direction of $\vec{u}$, $D_{\vec{u}} f(3,2)$.

We expect this number to be positive because the curve is pointing up as we walk along the curve (in dir of $\vec{u}$) from point $(3,2,\frac{25}{4})$.

We compute $D_{\vec{u}} f(3,2)$ using the dot product formula:

We computed this earlier

$$D_{\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle} f(3,2) = \langle f_x(3,2), f_y(3,2) \rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$$

$$= \left\langle \frac{2x}{4}\Big|_{(3,2)}, \; y\Big|_{(3,2)} \right\rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$$

$$= \langle \frac{2}{4}(3), 2 \rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$$

$$= \frac{1}{2}(3)\frac{\sqrt{2}}{2} + 2\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{4} + \sqrt{2} = \boxed{\frac{7\sqrt{2}}{4}}$$

(Additional examples)

Ex 2: Find the gradient $\nabla f(3,2)$

for $f(x,y) = x^2 + 2xy - y^3$,

Sol: $\nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y) \rangle$

$= \langle 2x + 2y + 0,\ 0 + 2x - 3y^2 \rangle$

$= (2x + 2y)\hat{i} + (2x - 3y^2)\hat{j}$

$\nabla f(3,2) = \langle 2(3) + 2(2),\ 2(3) - 3(2^2) \rangle$

$= \langle 6 + 4,\ 6 - 12 \rangle$

$= \langle 10, -6 \rangle$

$= 10\hat{i} - 6\hat{j}$

(Additional examples)

Ex 3: Let $f(x,y) = 3 - \frac{x^2}{10} + \frac{xy^2}{10}$.

a) Compute the gradient $\nabla f(3,-1)$:

$$f_x = 0 - \frac{2x}{10} + \frac{y^2}{10}, \quad f_y = 0 - 0 + \frac{x}{10} 2y = \frac{xy}{5}$$

$$= \frac{-2x + y^2}{10}$$

$$\nabla f(3,-1) = \langle f_x(3,-1), f_y(3,-1) \rangle$$

$$= \left\langle \frac{-2(3)+1}{10}, \frac{3(-1)}{5} \right\rangle = \boxed{\left\langle -\frac{5}{10}, -\frac{3}{5} \right\rangle}$$

b) Compute the directional derivative of f at $(3,-1)$ in the direction of the unit vector $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$:

$$D_{\vec{u}} f(3,-1) = \underbrace{\nabla f(3,-1)}_{\text{gradient}} \cdot \vec{u} = \left\langle -\frac{1}{2}, -\frac{3}{5} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = -\frac{1}{2\sqrt{2}} + \frac{3}{5\sqrt{2}} = \frac{-5+6}{10\sqrt{2}} = \boxed{\frac{1}{10\sqrt{2}}}$$

So the slope of the surface in the direction of $\left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$ is $\frac{1}{10\sqrt{2}}$

c) Compute the directional derivative of f at $(3,-1)$ in the direction of the (non-unit) vector $\vec{v} = \langle 3, 4 \rangle$:

Because $\vec{v}$ has length $\sqrt{3^2+4^2} = 5$, the unit vector in the direction of $\vec{v}$ is $\vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

$$\text{So } D_{\vec{u}} f(3,1) = \underbrace{\nabla f(3,-1)}_{\text{gradient}} \cdot \vec{u} = \left\langle -\frac{1}{2}, -\frac{3}{5} \right\rangle \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = -\frac{39}{50}.$$

So the slope of the surface in the direction of $\langle 3,4 \rangle$ is $\frac{-39}{50}$

From earlier

② You release a ball and let it roll. In which direction will it roll?

③ If you are hiking up a mountain, in what direction should you walk after each step if you want to follow the steepest path?

$$D_{\vec{u}} f(a,b) = \overbrace{\nabla f(a,b)}^{\text{the gradient}} \cdot \vec{u}$$

$\boxed{\text{Recall geometric def of dot product}}$ $= |\nabla f(a,b)| \underbrace{|\vec{u}|}_{\text{equal to } 1} \cos\theta$

$= |\nabla f(a,b)| \cos\theta$, where θ is the angle between $\nabla f(a,b)$ and $\vec{u}$.

So...

Thm

1) • $D_{\vec{u}} f(a,b)$ has its maximum value when $\cos\theta = 1$ (when $\theta = 0$)

• f has its greatest rate of increase at point $P(a,b)$ when $\nabla f(a,b)$ and $\vec{u}$ point in the same direction.

This rate of change is $|\nabla f(a,b)|$

• A vector that gives the direction of steepest ascent at point $P(a,b)$ is the gradient $\nabla f(a,b)$.

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2) • $D_{\vec{u}} f(a,b)$ has its minimum value when $\cos\theta = -1$ (when $\theta = \pi$).

• f has its greatest rate of decrease at point $P(a,b)$ when $\nabla f(a,b)$ and $\vec{u}$ point in opposite directions.

This rate of change is $-|\nabla f(a,b)|$

• A vector that gives the direction of steepest descent at point $P(a,b)$ is $-\nabla f(a,b) = \langle -f_x(a,b), -f_y(a,b) \rangle$.

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3) • $D_{\vec{u}} f(a,b) = 0$ when $\cos\theta = 0$ (when $\theta = \frac{\pi}{2}$).

• The directional derivative $D_{\vec{u}} f(a,b)$ is zero at point $P(a,b)$ when $\nabla f(a,b)$ and $\vec{u}$ are orthogonal.

• A vector $\vec{v}$ points in a direction of no change at point $P(a,b)$ if $\vec{v}$ is orthogonal to the gradient $\nabla f(a,b)$.

Ex 4: Consider $z = f(x,y) = 4 + x^2 + 3y^2$

a) $\nabla f = \langle 2x, 6y \rangle$

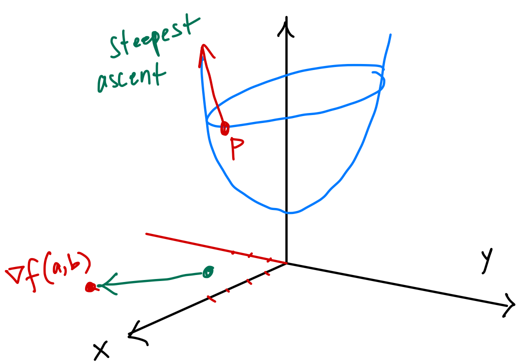
b) At point $P(\underset{a}{2}, \underset{b}{-\frac{1}{2}}, \underset{f(a,b)}{\frac{35}{4}})$, the gradient is $\nabla f(2,-\frac{1}{2}) = \langle 2x, 6y \rangle \Big|_{(2,-\frac{1}{2})} = \langle 4, -3 \rangle$

c) • So, at point $P(2, -\frac{1}{2}, \frac{35}{4})$, the vector $\langle 4,-3 \rangle$ or any positive scalar multiple of $\langle 4,-3 \rangle$ (like $\langle 1, -\frac{3}{4} \rangle$ or $\langle 40, -30 \rangle$) is a vector which gives the direction of steepest ascent.

• The rate of change in this direction is the magnitude of $\nabla f(2,-\frac{1}{2})$, which is $|\langle 4,-3 \rangle| = \sqrt{16+9} = 5$.

d) • The direction of steepest descent is given by $-\nabla f(2,-\frac{1}{2}) = \langle -4, 3 \rangle$ (or any positive scalar multiple)

• The rate of change is $-|\nabla f(2,-\frac{1}{2})| = -5$.



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→ (con't)

Ex 4: Consider $z = f(x,y) = 4 + x^2 + 3y^2$

a) $\nabla f = \langle 2x, 6y \rangle$

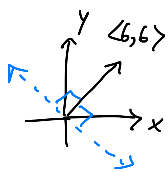
e) At point $P(\underset{a}{3}, \underset{b}{1}, \underset{f(a,b)}{16})$, in what direction(s) is there no change in the value of function f?

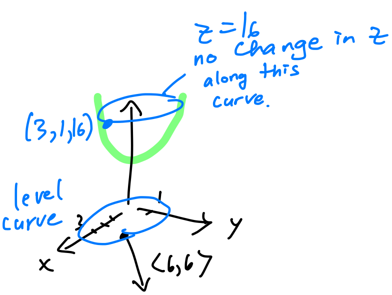
<u>Solution:</u> $\nabla f(3,1) = \langle 2x, 6y \rangle \Big|_{(3,1)} = \langle 6, 6 \rangle$

A vector perpendicular to $\langle 6,6 \rangle$ is $\langle -6, 6 \rangle$ or $\langle 6, -6 \rangle$

(To find this, I solved for $\langle 6,6 \rangle \cdot \langle \underbrace{1}_{\text{any nonzero number}}, \boxed{?} \rangle = 0$

$\Rightarrow \boxed{?} = -1$)





So f has zero change if we move in the direction of $\langle 6, -6 \rangle$ and $\langle -6, 6 \rangle$.

In terms of unit vectors, these are

$\frac{1}{\sqrt{2}} \langle -1, 1 \rangle$ and $\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$.

In general

<u>Thm</u> If $\underbrace{\nabla f(a,b) \neq \langle 0,0 \rangle}_{\text{the zero vector}}$, then the line tangent to the level curve of f at (a,b) is orthogonal to $\nabla f(a,b)$.

All def and thm work for functions $f(x,y,z)$.

Now $\nabla f \overset{\text{def}}{=} \langle f_x, f_y, f_z \rangle$