Reading HW for Sec 15.5
(Directional derivatives & the gradient)

Example 1, 2, 3

(Notes from previous Section 15.3)

<u>Thm/ Def</u>

If f_x and f_y exist on an open set containing (a,b)
and f_x and f_y are continuous at (a,b),
THEN f is <u>differentiable</u> at (a,b).

<u>Thm</u>

If f is differentiable at (a,b), then f is continuous at (a,b).

Contrapositive Statement:
If f is not continuous at (a,b),
then f is not differentiable at (a,b).

15.4 The Chain Rule

Recall Calc 1 Chain Rule:

If $y(u)$ is a function of u
& $u(t)$ is a function of t,

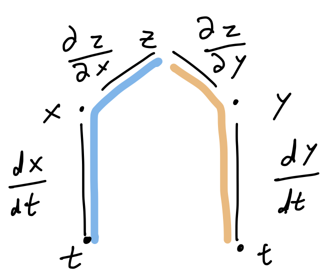
then $$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt}$$

New Chain Rule Thm (one independent variable t)

If $z = f(x,y)$ is a differentiable function of x and y,
& $x(t)$ and $y(t)$ are differentiable functions of t, then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Tree diagram:



Here, z is the dependent variable
t is the independent variable
x and y are intermediate variables

(• We write $\frac{dz}{dt}$ because z ultimately depends only on t

• We write $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ because they are partial derivatives

• We write $\frac{dx}{dt}, \frac{dy}{dt}$ because x and y depend only on t)

This theorem generalizes to functions of 3 or more intermediate variables. Ex: If $w = f(x,y,z)$, where x, y, z are functions of a single independent variable t, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

Ex 1: Let $z = x^2 - 3y^2 + 20$, where $x = 2\cos t$, $y = 2\sin t$.

Find $\frac{dz}{dt}$ and evaluate it at $t = \frac{\pi}{4}$.

Sol: $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$= 2x(-2\sin t) - 6y(2\cos t)$

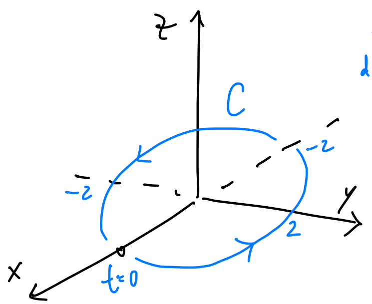
$= 2(2\cos t)(-2\sin t) - 6(2\sin t)(2\cos t)$

$= -8\cos t\sin t - 24\sin t\cos t$

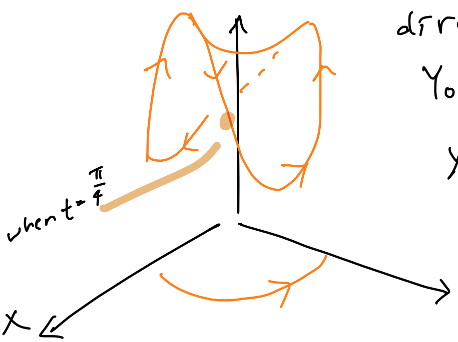
$= \boxed{-32\cos t\sin t}$

$\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{4}} = -32\cos\frac{\pi}{4}\sin\frac{\pi}{4} = -32\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = -8(2) = \boxed{-16}$

What is the meaning of $\frac{dz}{dt}$ at $t = \frac{\pi}{4}$?



The parametric equations $x = 2\cos t$, $y = 2\sin t$ for $0 \le t \le 2\pi$ describe a circle C of radius 2 in the xy-plane, in counterclockwise direction



Walk along the surface $z = x^2 - 3y^2 + 20$ directly above the circle C. Your path rises & falls as you walk. The rate of change of your elevation z w/ respect to time t is $\frac{dz}{dt}$.

Ex: when $t = \frac{\pi}{4}$, the point on the surface is $(x, y, z) = (\sqrt{2}, \sqrt{2}, 16)$.
At that point, z decreases at a rate of -16 as you walk on the surface above C.

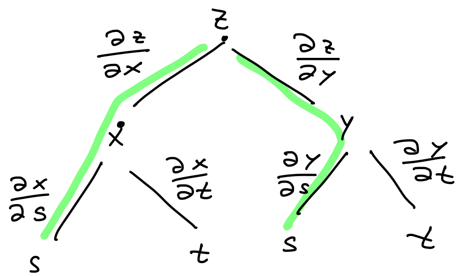
Thm Chain Rule (Two independent variables)

Let z be a differentiable function of x and y, where x & y are differentiable functions of s and t.

Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Tree diagram



(Note: these are all partial derivatives)

Ex 2: Let $z = \sin(2x)\cos(3y)$, where $x = s+t$ and $y = s-t$.

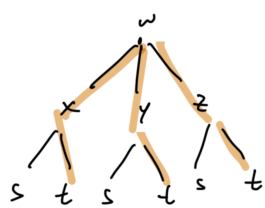
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \left(2\cos(2x)\cos(3y)\right) 1 + \left(-3\sin(2x)\sin(3y)\right) 1$$

(Additional Ex)

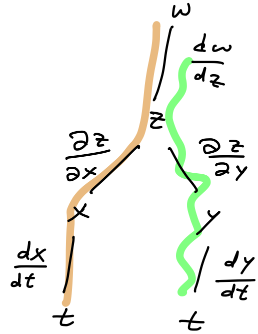
Ex 3: w is a function of x, y, z

x, y, z are functions of s and t



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

Ex 4: w is a function of z,
z is a function of x and y,
x and y are functions of t



Chain Rule: $$\frac{dw}{dt} = \underbrace{\frac{dw}{dz}\frac{\partial z}{\partial x}\frac{dx}{dt}} + \underbrace{\frac{dw}{dz}\frac{\partial z}{\partial y}\frac{dy}{dt}}$$

$$= \frac{dw}{dz}\underbrace{\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)}_{\frac{dz}{dt}}$$

(Additional Ex)

Ex 5 (Implicit Differentiation):

$\sin(xy) + \pi y^2 = x$ Find $\frac{dy}{dx}$.

Sol: $\underbrace{\sin(xy) + \pi y^2 - x}_{} = 0$

let $F(x,y) := \sin(xy) + \pi y^2 - x$

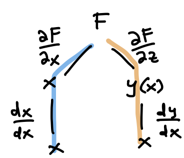
So the original eq is $F(x,y) = 0$

To find $\frac{dy}{dx}$, Think of y as a function of x: $F(x, y(x)) = 0$

differentiate both sides with respect to x. Think of x as the independent variable:

(From the 1st Chain Rule Thm)

$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{1} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$



$$\left(y\cos(xy) - 1\right)1 + \left(x\cos(xy) + 2\pi y\right)\frac{dy}{dx} = 0$$

$$\left(x\cos(xy) + 2\pi y\right)\frac{dy}{dx} = -y\cos(xy) + 1$$

$$\boxed{\frac{dy}{dx} = \frac{-y\cos(xy) + 1}{x\cos(xy) + 2\pi y}}$$