Reading HW:

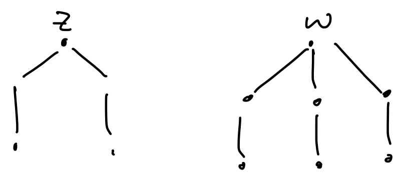
Sec 15.4 Chain Rule

Read ...

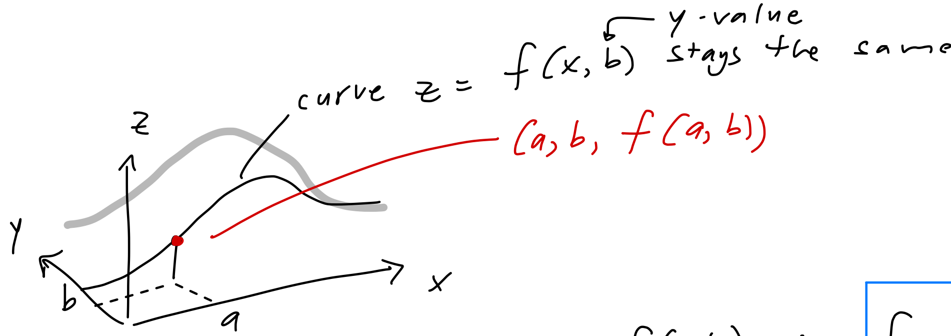
* the first theorem (Chain Rule w/ only 1 independent variable)

* Example 1

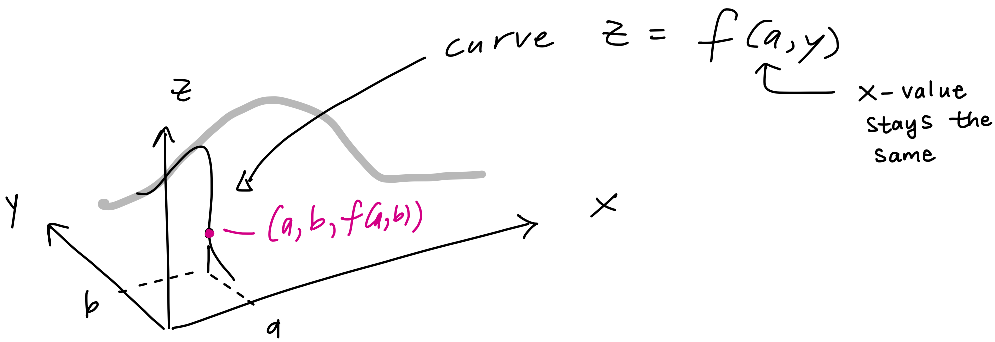
* The first two figures ("Tree diagrams")



15.3 Partial derivatives



"slope of the curve" $z = f(x,b)$ is $\boxed{f_x(a,b)}$, the partial derivative of f with respect to x at point (a,b). Other notation: $\frac{\partial f}{\partial x}(a,b)$ or $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$



"slope of the curve" $z = f(x,b)$ is $\boxed{f_y(a,b)}$, the partial derivative of f with respect to y at point (a,b).

Other notation: $\frac{\partial}{\partial y}(a,b)$ or $\left.\frac{\partial f}{\partial y}\right|_{(a,b)}$

To compute $\frac{\partial f}{\partial x}$, treat y like a constant
" $\frac{\partial f}{\partial y}$, " x like a constant

Ex: • $\frac{\partial}{\partial x}(5x^2 y) = 5y \frac{\partial}{\partial x} x^2 = 5y \; 2x$
$= \boxed{10xy}$

treat y as a constant

• $\frac{\partial}{\partial y}(5x^2 y) = 5x^2 \frac{\partial}{\partial y} y = \boxed{5x^2}$

treat x as constant 1

• $\frac{\partial}{\partial x} y^4 = 0$

Ex 2:

both constants wrt y

• $f(x,y) = x^3 - y^2 + 4$

• $\frac{\partial}{\partial y} = -2y$

• $f_y(2,-4) = \frac{\partial}{\partial y}\Big|_{(2,-4)} = (-2y)\Big|_{(2,-4)} = -2(-4) = \boxed{8}$

Ex 3: • $f(x,y) = \sin(xy)$

constant wrt x

• $\frac{\partial f}{\partial x} = \boxed{y \cos(xy)}$

Second-order partial derivatives

Notation 1:	Notation 2:	Say:	or say:
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$	$(f_x)_x = f_{xx}$	"d squared f dx squared"	"f xx"
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$	$(f_y)_y = f_{yy}$	"d squared f dy squared"	or "f-y-y"
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$	$(f_y)_x = f_{yx}$	Say "f-y-x"	
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$	$(f_x)_y = f_{xy}$	Say "f-x-y"	

(The order matters in the mixed partial derivatives f_{xy}, f_{yx})

Ex 4: $f(x,y) = 3x^4 y - 2xy + 5xy^3$

(First) partial derivatives

$$\frac{\partial f}{\partial x} = \boxed{12x^3 y - 2y + 5y^3}$$

$$\frac{\partial f}{\partial y} = \boxed{3x^4 - 2x + 15xy^2}$$

Second partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \boxed{36x^2 y} + 0 + 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \boxed{12x^3 - 2 + 15y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 0 + 0 + \boxed{30xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \boxed{12x^3 - 2 + 15y^2}$$

Notice that "the two mixed partial derivatives" $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are equal.

Thm (Equality of mixed partial derivatives)

If f_{xy} and f_{yx} are continuous on an open set D of $\mathbb{R}^2$,

then $f_{xy} = f_{yx}$ at all points of D.

Ex 5 Let $f(x,y,z) = e^{-xy}\cos z$,

a function of three variables.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{-xy} \cdot \underbrace{\cos z}_{\text{constant}}\right) = \boxed{-ye^{-xy}\cos z}$$

y is constant

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(e^{-xy}\cos z\right) = \boxed{e^{-xy}(-\sin z)}$$

constant

Applications of partial derivatives: investment fund, mathematical modeling of population, health care.

Ex 6 (Ideal Gas Law) (Similar to MML #10)

(molecules don't attract or repel each other, & molecules have no size)

theoretical

The ideal gas follows the Law $PV = kT$,

where P is pressure

V is volume

T is temperature (in kelvins, which is always positive)

$k>0$ is a constant depending on the amount of gas.

a) Determine the rate of change of the pressure with respect to the volume at a constant temperature. Interpret the result.

Sol: Express the pressure as a function of volume and temperature:

$$P(V, T) = \frac{kT}{V}$$

$$\frac{\partial P}{\partial V} = kT \frac{\partial}{\partial V}\left(V^{-1}\right) = kT\left(-V^{-2}\right) = -\frac{kT}{V^2}$$

Interpretation: Since k, V, T are positive, we see that $\frac{\partial P}{\partial V} < 0$. This means the pressure is a decreasing function of volume (at a constant temp).

(more volume, less pressure)

b.) Rate of change of the pressure with respect to the temperature at constant volume? Interpretation?

Sol: $\frac{\partial P}{\partial T} = \frac{k}{V}$

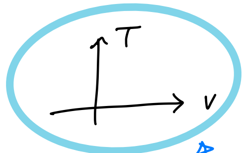
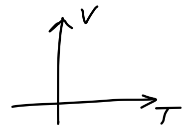
Interpretation: $\frac{\partial P}{\partial T} > 0$. This means the pressure is an increasing function of temperature (at constant volume).

(higher temperature, more pressure).

c.) Draw several level curves, & interpret the results.

The level curves of the function $P(V,T) = k\frac{T}{V}$ are curves in the VT-plane with equations

$\boxed{k\frac{T}{V} = P_0}$ where P_0 is constant in the range/image of $P(V,T)$, (always pos) meaning P_0 is in $(0, \infty)$

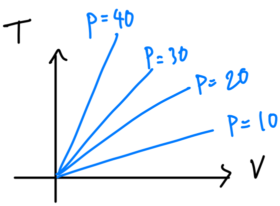
We can draw the VT-plane as  or 

I pick this

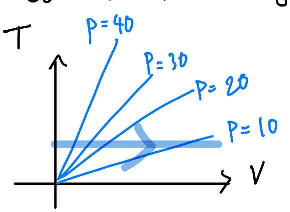
Rewrite the level curve eq so that T is on one side:

$\boxed{T = \frac{1}{k}P_0 V}$

We get lines w/ slope $\frac{1}{k}P_0$ (always positive) for V in $(0, \infty)$:



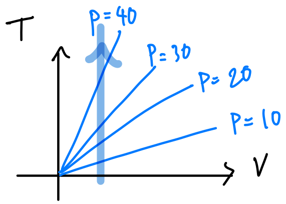
* The fact that $\frac{\partial P}{\partial V} < 0$ means that, if we hold T fixed and move in the direction of increasing V on a horizontal line, we cross level curves corresp. to decreasing pressures.



on a line of constant T, P decreases as V increases.

(Additional notes)

* The fact that $\frac{\partial P}{\partial T} > 0$ means that, if we hold V fixed and move in the direction of increasing T on a vertical line, we cross level curves corresp. to increasing pressures.



on a line of constant V, P increases as T increases.

(Notes for next class)

Thm/Def

If f_x and f_y exist on an open set containing (a,b) and f_x and f_y are continuous at (a,b),

THEN f is differentiable at (a,b).

Thm

If f is differentiable at (a,b), then f is continuous at (a,b).

Contrapositive statement:

If f is not continuous at (a,b), then f is not differentiable at (a,b).