15.2 Limits & continuity (for functions of two variables)

(We revisit Calc I concepts, starting from limits & continuity)

Def A function $f(x,y)$ <u>has a limit</u> L as $P(x,y) \to P_0(a,b)$

a specific point

"approaches"

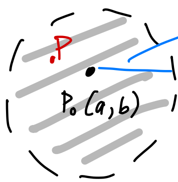
if $|f(x,y) - L|$ can be made <u>arbitrarily small</u> (as small as you want)

for all points P in a sufficiently small disk centered at $P_0(a,b)$.

i.e. if, given any small positive number ϵ (epsilon), we can find δ (delta)

such that:

P being in the disk

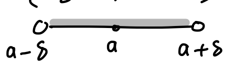


(meaning the distance between P and P_0 is less than δ)

implies $|f(x,y) - L| < \epsilon$.

Warning: • In Calc I, this 1-dimensional "disk" is

the interval $(-\delta+a,\ a+\delta)$



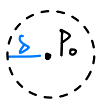
, so we only needed to

check whether $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist and are equal.

x approaches a from the right; x approaches a from the left

(There are only 2 paths to get to a, from the left & from the right)

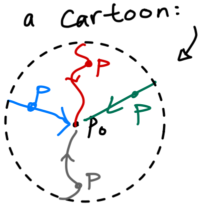
• Now, in Calc III,



is in $\mathbb{R}^2$, so the limit exists only if

$f(x,y)$ approaches L as P approaches P_0 along <u>all</u> possible paths (!!)

Here's a cartoon:



(Here, I sketched just four (out of <u>infinitely</u> many) paths $P \to P_0$)

Note:

* In order for the limit of f to exist at $P_0(a,b)$, we need $f(x,y) \to L$ as $P \to P_0$ for every path.
* If there are two paths $P \to P_0$ that produces different limits, the limit $\lim_{(x,y)\to P_0} f(x,y)$ DOESN'T EXIST.

<u>Thm</u> (limits for constant & linear functions)

1. Constant function $f(x,y) = C$:
$$\lim_{(x,y)\to(a,b)} f(x,y) = C$$

1. Ex: $\lim_{(x,y)\to(1,2)} 3 = 3$

2. Linear function $f(x,y) = x$:
$$\lim_{(x,y)\to(a,b)} f(x,y) = a$$

2. Ex: $\lim_{(x,y)\to(4,5)} x = 4$

3. Linear function $f(x,y) = y$:
$$\lim_{(x,y)\to(a,b)} f(x,y) = b$$

3. Ex: $\lim_{(x,y)\to(4,5)} y = 5$

Thm (Limit laws)

Suppose $\lim_{(x,y)\to(a,b)} f(x,y) = L$, $\lim_{(x,y)\to(a,b)} g(x,y) = M$, and

L, M are real numbers

c, d are real numbers.

(i) $\lim_{(x,y)\to(a,b)} \left[c\, f(x,y) + d\, g(x,y)\right] = cL + dM$ (linear combination)

(this means limit is a "linear" operation)

(ii) $\lim_{(x,y)\to(a,b)} f(x,y)\, g(x,y) = LM$ (product)

(iii) $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ if $M \neq 0$ (quotient)

(iv) If n is a positive integer, $\lim_{(x,y)\to(a,b)} \left(f(x,y)\right)^{\frac{1}{n}} = L^{\frac{1}{n}}$

$\sqrt[n]{f(x,y)}$ $\qquad$ $\sqrt[n]{L}$

Ex 1: Evaluate $\lim_{(x,y)\to(2,8)} \left(3x^2y + \sqrt{xy}\right)$

Sol: $\lim_{(x,y)\to(2,8)} x = 2$, $\lim_{(x,y)\to(2,8)} y = 8$

$$\lim_{(x,y)\to(2,8)} 3x^2y = 3\,(2)^2\, 8 = 96$$

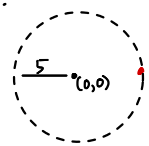
$$\lim_{(x,y)\to(2,8)} \sqrt{xy} = \sqrt{2\cdot 8} = \sqrt{16} = 4$$

+

$$\boxed{\lim_{(x,y)\to(2,8)} \left(3x^2y + \sqrt{xy}\right) = 100}$$

Def Let R be a region in $\mathbb{R}^2$

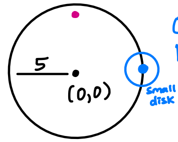
Ex:



This point $Q(5,0)$ is a boundary point of R although not in R

Here R is the (open) disk $x^2+y^2 < 5^2$

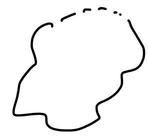
Ex:
$P(0, 4.99)$ is an interior point



$Q(5,0)$ is a boundary point

Here R is the disk $x^2+y^2 \le 5^2$

Ex:



R

1. A point P of R is an interior point if it's possible to find a disk centered at P that's contained in R

2. A point Q of R is a boundary point if Q lies "on the edge" of R in the sense that every disk centered at Q contains a point in R and a point outside of R.

Note: A boundary point of a region R needs not lie in R.

Ex: The boundary points of the disk $x^2+y^2 < 25$ lie on the circle $x^2+y^2 = 25$.

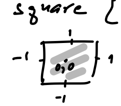
The interior points of the disk $x^2+y^2 < 25$ are all points of this disk.

3. A region is open if it consists entirely of interior points.

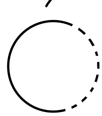
4. — " — closed if it contains all its boundary points

Ex: The disk $x^2+y^2 < 25$ is open. The entire $\mathbb{R}^2$ is open.

The disk $x^2+y^2 \le 25$ is closed. The square $\{(x,y): |x| \le 1, |y| \le 1\}$



is closed.

The region  is neither closed nor open.

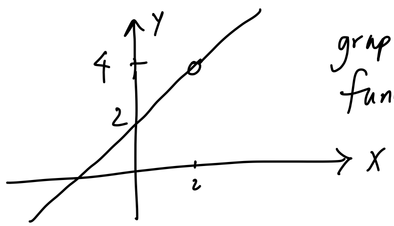
Note: Will later generalize this concept to 3D regions: balls, boxes, pyramids.

Limits at boundary points

Recall Calc 1: limit may exist even if the function is not defined at a point.

Ex: $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} x+2 = 4$

equals $x+2$ for all numbers except $x=2$



graph of function $f(x) = \frac{x^2-4}{x-2}$

The domain of $f(x)$ is $(-\infty, 2) \cup (2, \infty)$

and $x=2$ is a boundary point of the domain.

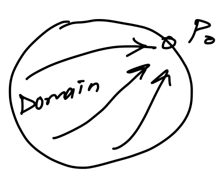
Now for $f(x,y)$ (function of two vars), suppose $P_0(a,b)$ is a boundary point of the domain of $f(x,y)$.

Even if P_0 is not in the domain of $f(x,y)$,

the limit $\lim_{(x,y) \to (a,b)} f(x,y)$ exists IF

$f(x,y)$ approaches the same value L

as $(x,y) \to (a,b)$ ALONG ALL PATHS THAT LIE IN THE DOMAIN of $f(x,y)$.



Ex (pg 933): Let $f(x,y) = \dfrac{x^2 - y^2}{x - y}$.

Its domain is $\{(x,y) : x \neq y\}$.

When $x \neq y$, we can cancel the factor $(x-y)$ from the numerator & denominator:

$$f(x,y) = \frac{x^2 - y^2}{x - y} = \frac{(x-y)(x+y)}{x-y} \overset{\text{if } x \neq y}{=} x+y$$

Then the graph of $f(x,y)$ is $z = x + y$
$0 = x + y - z$

(plane w/ a normal vector $\langle 1, 1, -1 \rangle$ passing through $(0,0,0)$)
BUT with the points corresponding to $x = y$ removed.

Consider $P_0(4,4)$. This is a boundary point of the domain of f but doesn't lie in the domain (because $x = y = 4$)

For $\displaystyle\lim_{(x,y)\to(4,4)} \frac{x^2 - y^2}{x-y}$ to exist,

$f(x,y)$ must approach the same value along EVERY path to $P_0(4,4)$ that lie in the domain of f, that is, every path to $P_0(4,4)$ that don't intersect the line $y = x$.

$$\lim_{(x,y)\to(4,4)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(4,4)} \frac{(x+y)(x-y)}{x-y} \overset{\text{every path that lies in domain of } f}{=} \lim_{(x,y)\to(4,4)} (x+y)$$

$$= 4+4 = \boxed{8}$$

Ex 3: (See also MML #4)

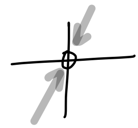
Investigate $f(x,y) = \dfrac{(x+y)^2}{x^2+y^2}$ as $(x,y) \to (0,0)$.

Ans: Domain of f is $\{(x,y) : (x,y) \neq (0,0)\} = \underline{\mathbb{R}^2 - \{(0,0)\}}$

entire xy-plane minus one point, the origin.

Note: $(0,0)$ is a bdry pt outside the domain.

* First, let (x,y) approach $(0,0)$ along the line $y = 3x$:

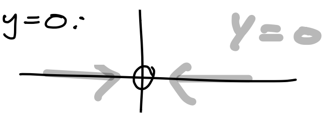


$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=3x}} \frac{(x+y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{(4x)^2}{x^2+(3x)^2} = \lim_{x\to 0} \frac{4^2x^2}{10x^2} = \lim_{x\to 0} \frac{4^2}{10} = \boxed{\frac{4^2}{10}}$$

sub $y := 3x$

this function is $\frac{4^2}{10}$ when $x \neq 0$

* Next, let (x,y) approach $(0,0)$ along the line $y=0$:



$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{(x+y)^2}{(x^2+y^2)} = \lim_{x\to 0} \frac{x^2}{x^2} = \boxed{1}$$

$y := 0$

* Conclusion:

Because $f(x,y)$ has different limits ($\frac{16}{10}$ and 1) along two different paths in the domain of f as (x,y) approaches $(0,0)$,

$$\boxed{\lim_{(x,y)\to(0,0)} f(x,y) \quad \text{DOES NOT EXIST.}}$$

Def $f(x,y)$ is <u>continuous</u> at point (a,b) if

1. f is defined at (a,b)
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Thm Composition of continuous functions is continuous.

Ex: (MML #3)

$\ln(u)$ is continuous for u in $(0,\infty)$

$\sqrt[3]{xy}$ is continuous for (x,y) s.t $x>0, y>0$.

$\ln\sqrt[3]{xy}$ is continuous

$$\lim_{(x,y)\to(e^9,8)} \ln\left(\sqrt[3]{xy}\right) = \ln\left(\sqrt[3]{e^9 8}\right) = \ln(2e^3)$$

$$= \ln(2) + \ln(e^3)$$

$$= \boxed{\ln 2 + 3}$$