15.1 Graphs & level curves

Def: A function f of 2 variables can be written in the "explicit form" $z = f(x,y)$.

The domain D of f is the set of all possible inputs (x,y) in $\mathbb{R}^2$

each of x & y is a number

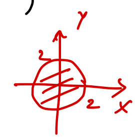
The image (or range) of f is the set of all possible outputs z in $\mathbb{R}$.

Ex: • The domain D of the function $g(x,y) = \sqrt{4 - x^2 - y^2}$ is the set of ordered pairs (x,y) for which

$4 - x^2 - y^2 \geq 0$ (because of the square root $\sqrt{\ }$)

$\Leftrightarrow 4 \geq x^2 + y^2$

So $D = \{ (x,y) : x^2 + y^2 \leq 4 \}$ the closed disk w/ radius 2



• The domain of $f(x,y) = \sin xy$ is $\mathbb{R}^2$

$\{ (x,y) : x$ is any number, y is any number $\}$

because $\sin t$ is defined for any number t

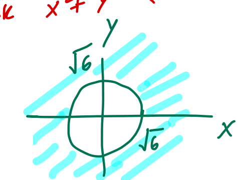
Additional Ex: (MML #5)

• Consider the function $h(x,y) = \sqrt{-6 + x^2 + y^2}$

• The domain of h is

$$\{ (x,y) : \underbrace{-6 + x^2 + y^2 \geq 0}_{x^2 + y^2 \geq 6} \}$$

$\mathbb{R}^2$ minus the open disk $x^2 + y^2 < 6$

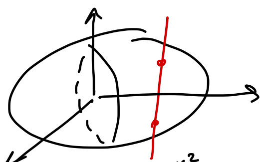


• The range / image of h is

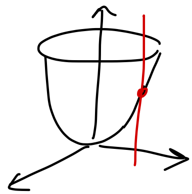
$\{ -6 + x^2 + y^2 : -6 + x^2 + y^2 \geq 0 \} = \{$ numbers bigger or equal to $0 \} = [0, \infty)$

Def: • The graph of $f(x,y)$ is the set of points (x,y,z) in $\mathbb{R}^3$ that satisfy the equation $z = f(x,y)$, i.e. for each point (x,y) in the domain of f, the point $(x, y, f(x,y))$ lies on the graph of f.

• A graph is the graph of a function of 2 variables if and only if it passes the "vertical line tes".



An ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ fails the vertical line test, so it's not the graph of a function



An elliptic paraboloid $z = ax^2 + by^2$ is the graph of a function $f(x,y) = ax^2 + by^2$

Ex: $g(x,y) = x^2 + y^2$

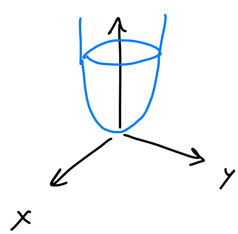
Domain of g is $\mathbb{R}^2$

Range of g is $[0, \infty)$ because $x^2 + y^2 \geqslant 0$

Graph of g is the graph for the eq $z = x^2 + y^2$

This is an elliptic paraboloid that opens upward.

w/ vertex $(0,0,0)$



Ex a) Find the domain & range (image) of the function $f(x,y) = 2x + 3y - 12$

b) then identify / sketch the graph.

Sol:
a)
- The domain of f is $\mathbb{R}^2$
- The range/image of $\mathbb{R}$

b) Sketch $z = 2x + 3y - 12$:

$$12 = 2x + 3y - z$$

This is of the form $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$, so this surface is a plane with normal vector $2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ (see Sec 13.5)

ended here Week 4 Friday

- Find the x-intercepts:

Set the other variables to 0:

$y = z = 0 \Rightarrow 12 = 2x$

$6 = x$

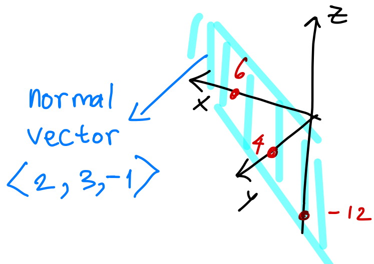
$\boxed{(6,0,0)}$

- Find the y-intercepts:

Set $x = y = 0$: $12 = 3y$

$4 = y$

$\boxed{(0,4,0)}$



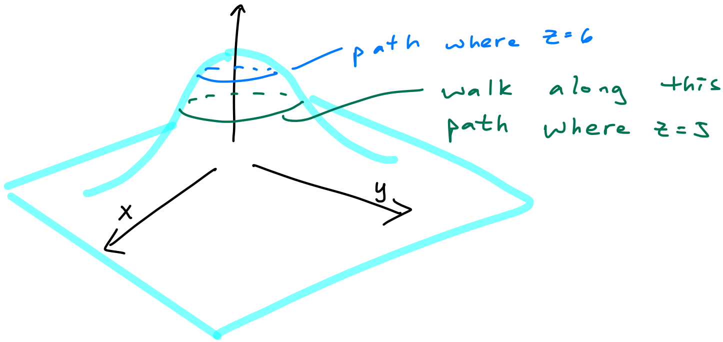
- Find the z-intercepts:

Set $x = y = 0$: $12 = -z$

$-12 = z$

$\boxed{(0,0,-12)}$

Idea of a contour curve: Consider a surface $z = f(x,y)$ which is a smooth mountain



Walk along a path on which your elevation has the constant value $z = 5$. This is an example of a contour curve.

Def
- A contour curve is the intersection of a surface and the horizontal plane $z = z_0$. The equations are:

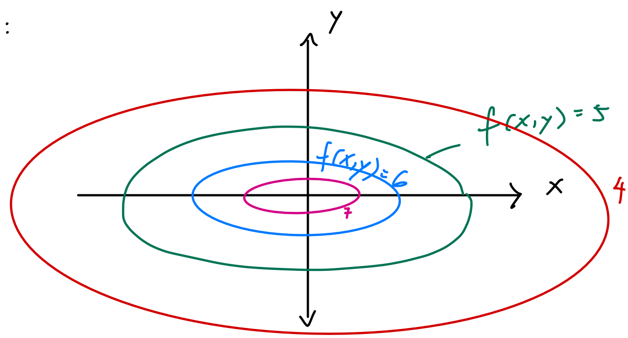
$$z = f(x,y) \quad \text{AND} \quad z = z_0.$$

(Note: a contour curve lives in $\mathbb{R}^3$)

- A level curve is the projection of a contour curve onto the xy-plane. The equation is: $f(x,y) = z_0$. Ex: $f(x,y) = 5$.

(Note: a level curve lives in $\mathbb{R}^2$, on the xy-plane)

Level curves of f:



In general, closely spaced curves means rapid changes in elevation
widely spaced curves means slow changes in elevation.

Application: topographic map

Ex 3 (a): Find & sketch several level curves of $f(x,y) = y - x^2 - 1$.

Label at least two level curves w/ their z-values.

Sol: Image/range of f is $\mathbb{R}$

The level curves (by def) are described by equations

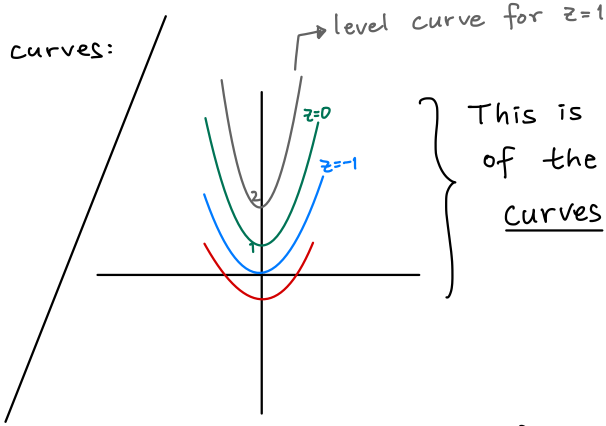
$\boxed{y - x^2 - 1 = z_0}$ where z_0 is a constant in $\underbrace{\text{the image/range of } f}_{\text{which in this example is } \mathbb{R}}$

$\boxed{y = x^2 + 1 + z_0}$

↳ These are parabolas in the xy-plane

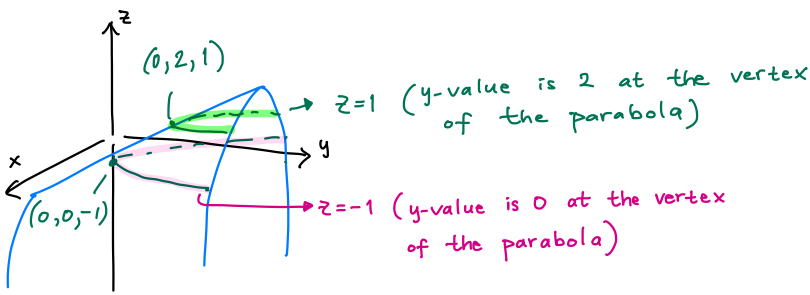
Write several level curves:

z_0	level curve
0	$y = x^2 + 1$
-1	$y = x^2$
1	$y = x^2 + 2$



This is a sketch of the level curves

Note: With desmos.com/3d, we can graph $z = y - x^2 - 1$:



Note:

These parabolas in $\mathbb{R}^3$ are the intersection of the surface & the horizontal planes $z = z_0$. They are the contour curves of $f(x,y) = y - x^2 - 1$.

Ex 4: Find & sketch several level curves of $f(x,y) = 2 + \sin(x-y)$.

Sol: Image of f? $1 \le z \le 3$

Possible values of $\sin(x-y)$ are $[-1, 1]$, so

possible values of $2+\sin(x-y)$ are $[-1+2, 1+2] = [1, 3]$

The level curves (by def) are described by equations

$$2+\sin(x-y) = z_0 \quad \text{where} \quad 1 \le z_0 \le 3.$$

when $z_0 = 2$, the level curve is $2+\sin(x-y) = 2$

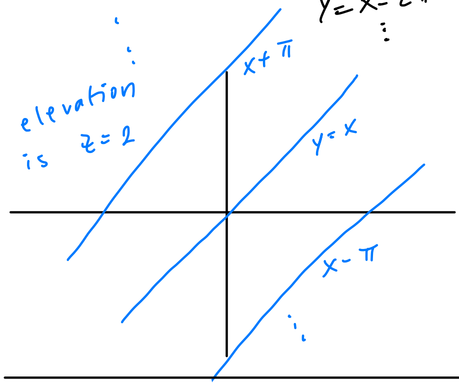
$\sin(x-y) = 0$

$x-y = \ldots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$

So, when $z_0 = 2$, the level curve is an infinite collection of

$$\text{lines} \quad \left.\begin{array}{l} \vdots \\ y = x+\pi \\ y = x \\ y = x-\pi \\ y = x-2\pi \\ \vdots \end{array}\right\} \quad \boxed{y = x - k\pi \quad \text{where } k \text{ is an integer}}$$

This describes the level curve for $z_0 = 2$



when $z_0 = 1$ (the minimum value for z),

the level curve satisfies $\sin(x-y) = -1$

$x-y = -\frac{\pi}{2} + 2k\pi$ where k is any integer

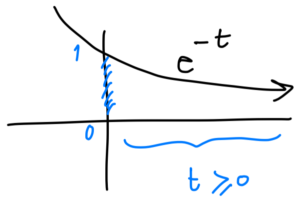
(Again, we have an infinite collection of lines w/ slope 1)

(Extra example)

Ex 3(b) Find & Sketch the level curves of

$$f(x,y) = e^{-x^2-y^2}$$

Sol: Range of f is $\{e^{-t} : t \geq 0\}$ because $x^2+y^2 \geq 0$

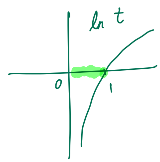


so the range of f is $(0,1]$

The level curves satisfy the equation $e^{-x^2-y^2} = z_0$ where $0 < z_0 \leq 1$.

Take natural logarithm of both sides: $-x^2-y^2 = \ln z_0$

$$x^2+y^2 = \underbrace{-\ln z_0}_{\text{note this is non negative because } z_0 \text{ is in } (0,1]}.$$

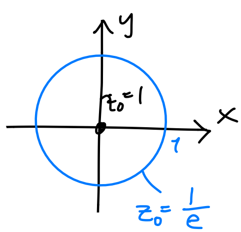


So the level curves are circles (except when $z_0 = 1$; in this case the level curve is the point $(0,0)$).

- When $z_0 = 1$, the level curve satisfies $x^2+y^2 = -\ln(1) = 0$. The only solution to this equation is $(0,0)$.

- When $z_0 = \frac{1}{e}$, the level curve satisfies $x^2+y^2 = -\ln(e^{-1})$
$$x^2+y^2 = 1$$
which is the unit circle centered at $(0,0)$.

These two level curves:



The surface $z = e^{-x^2-y^2}$