(Fri) Exam 1: up to this Sec, 14.5

Reading HW:

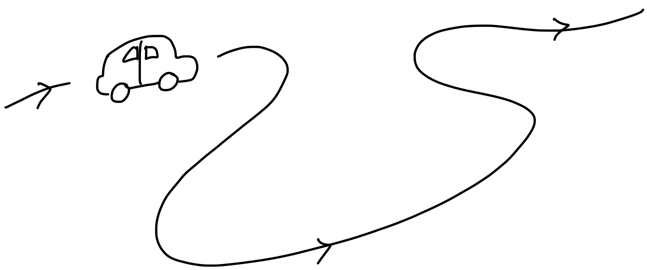
Sec 15.1 : Graphs & level curves

* New def of domain

* Example 2, 3, 4

No School this Monday

14.5 Curvature & normal vectors



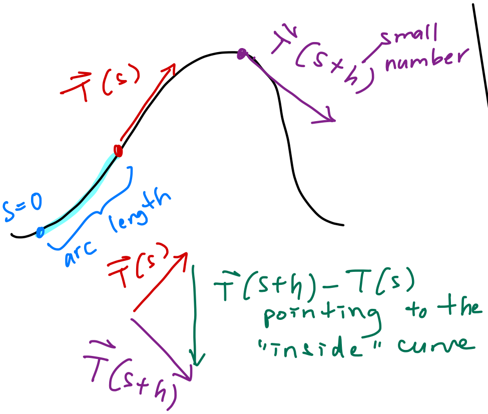
(Imagine driving a car along a winding mountain road)

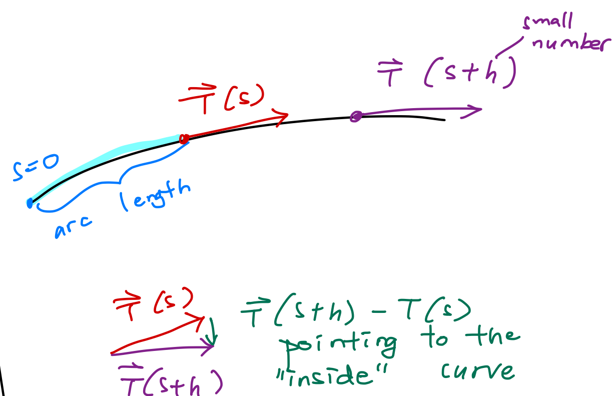
Part I

Curvature is the rate at which the car changes direction.

Recall $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$ is the unit tangent vector (from Sec 14.2)

Length of $\vec{T}$ is always 1, so the only way $\vec{T}$ can change is through a change in direction.





If moving forward by a bit along the curve corresponds to a large change in the direction of $\vec{T}$: say the curve has a large curvature

If moving forward by a bit along the curve corresponds to a small change in the direction of $\vec{T}$, say the curve has a small curvature

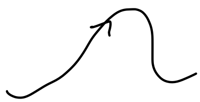
Def The <u>curvature</u> of a smooth curve $\vec r(t)$
meaning $\vec v(t)$ is never the zero vector

is $\kappa(s) = \left|\dfrac{d\vec T}{ds}\right|$, where s is arc length

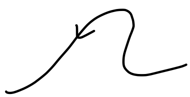
Kappa

Note:

1) Curvature depends only on the shape of the curve, not the orientation.

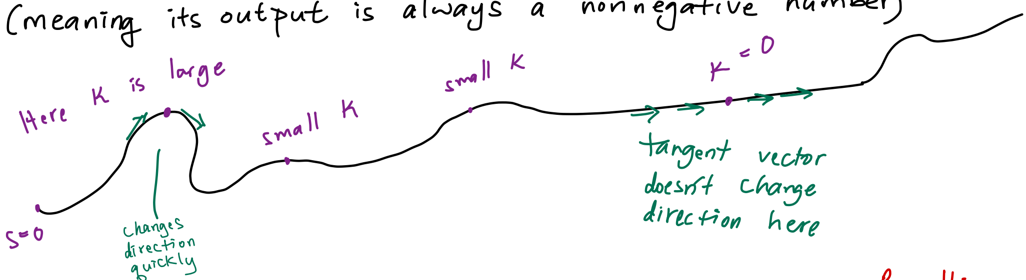


and



have the same curvature.

2) $\kappa(s)$ is a nonnegative scalar-valued function (meaning its output is always a nonnegative number)



$$\textbf{Thm}\quad \kappa(t) = \frac{1}{|\vec v|}\left|\frac{d\vec T}{dt}\right| = \frac{1}{|r'(t)|}\,|\vec T'(t)|$$

Recall:
If we have arc-length param, $|\vec v| = 1$ always

Why? $\dfrac{d\vec T}{dt} = \dfrac{d\vec T}{ds}\cdot\dfrac{ds}{dt}$ Chain Rule

$$\frac{\left(\frac{d\vec T}{dt}\right)}{\left(\frac{ds}{dt}\right)} = \frac{d\vec T}{ds}\cdot\cancel{\frac{ds}{dt}}$$ Divide by $\dfrac{ds}{dt} = |\vec v|$

$$\frac{1}{|\vec v|}\left(\frac{d\vec T}{dt}\right) = \frac{d\vec T}{ds} \Rightarrow \kappa(s) \underset{def}{=} \left|\frac{d\vec T}{ds}\right| = \frac{1}{|\vec v|}\left|\frac{d\vec T}{dt}\right|.$$

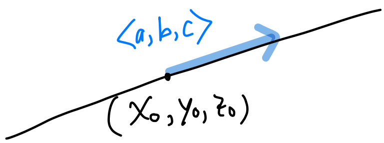
EX 1 (MML #8)

Recall the line passing through point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\langle a, b, c\rangle$ can be described by

$$\vec{r}(t) = \langle x_0 + at,\ y_0 + bt,\ z_0 + ct\rangle \quad \text{for } -\infty < t < \infty.$$

(see Sec 14.1 Example 1)

Then $\kappa(t)$ intuitively should be 0 because the unit tangent vector should always be parallel to $\langle a, b, c\rangle$



Let's verify that our intuition is correct:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle a, b, c\rangle}{\sqrt{a^2+b^2+c^2}}$$

derivative divided by its length

which is a constant vector (every component is a number)

so $\dfrac{d\vec{T}(t)}{dt} = \vec{0}$ the zero vector for all t

so $\kappa(t) = \dfrac{1}{|\vec{v}|}\left|\dfrac{d\vec{T}}{dt}\right| = \dfrac{1}{|\vec{v}|}|\vec{0}| = 0$ for all t

Punch line: the curvature of a straight line is zero always

There is another formula for computing curvature, using cross product:

- Let $\vec{a}$ be the acceleration of the object w/ position $\vec{r}(t)$. Then $\vec{a} = \dfrac{d\vec{v}}{dt}$

$$\boxed{K(t) = \frac{|\vec{v}(t) \times \vec{a}(t)|}{|\vec{v}(t)|^3}}$$

Ex 3 Find the curvature of the parabola
$$\vec{r}(t) = \langle t, t^2 \rangle, \quad -\infty < t < \infty.$$

Sol: $\vec{v}(t) = \vec{r}\,'(t) = \langle 1, 2t \rangle$

$$|\vec{v}|^3 = \left[1^2 + (2t)^2\right]^{\frac{3}{2}} = \left[1 + 4t^2\right]^{\frac{3}{2}}$$

$$\vec{a}(t) = \vec{v}\,'(t) = \langle 0, 2 \rangle$$

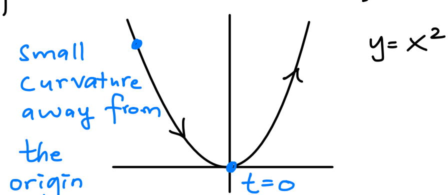
$$\vec{v}(t) \times \vec{a}(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = 0\,\hat{\imath} - 0\,\hat{\jmath} + (2-0)\,\hat{k} = \langle 0, 0, 2 \rangle$$

$$|\vec{v}(t) \times \vec{a}(t)| = \sqrt{2^2} = 2$$

So $\boxed{K(t) = \dfrac{2}{\left[1+4t^2\right]^{\frac{3}{2}}}}$

Remark: Curvature is biggest when the denominator is small $(t=0)$.
As $t \to \infty$ and $t \to -\infty$, curvature approach 0.

parabola $\vec{r}(t) = \langle t, t^2 \rangle$, $t \in (-\infty, \infty)$

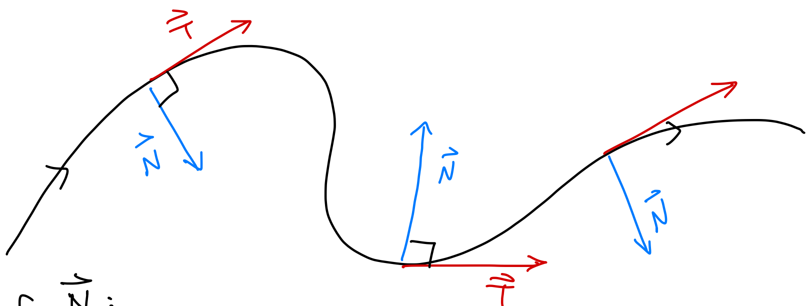


Part II Principal unit normal vector

Let $\vec{r}(t)$ be a smooth curve.

Idea: • curvature $\underbrace{\kappa(t)}_{\text{a number}} \overset{\text{def}}{=} \left| \frac{d\vec{T}}{ds} \right|$ tells us how fast the curve turns.

• The principal unit normal vector $\vec{N}$ is the direction $\frac{d\vec{T}}{ds}$ in which the curve turns, divided by its magnitude.



Properties of $\vec{N}$:

1) $|\vec{T}| = |\vec{N}| = 1$ always because both $\vec{T}$ and $\vec{N}$ are unit vectors.

2) $\vec{T} \cdot \vec{N} = 0$ always because they are perpendicular:

* $\vec{T}$ is tangent to the curve (in direction of curve)
* $\vec{N}$ is normal (perpendicular) to the curve and points to the "inside" of the curve.

Def The <u>principal unit normal vector</u> at point P on the curve (at which $\kappa(t) \neq 0$) is the vector

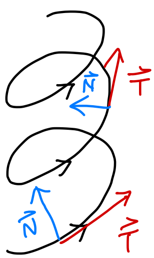
Curvature

$$\boxed{\vec{N}(t) = \frac{\left(\frac{d\vec{T}}{dt}\right)}{\left|\frac{d\vec{T}}{dt}\right|}}$$

evaluated at the value of t corresponding to P.

Ex 5 (Helix)

Let $\vec{r}(t) = \langle 3\cos t,\ 3\sin t,\ 4t \rangle$, for $-\infty < t < \infty$.



$$\vec{v}(t) = \vec{r}\,'(t) = \langle -3\sin t,\ 3\cos t,\ 4 \rangle$$

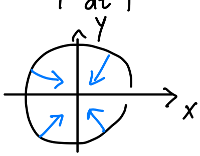
$$|v(t)| = \sqrt{3^2\sin^2 t + 3^2\cos^2 t + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\text{So } \vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\langle -3\sin t,\ 3\cos t,\ 4 \rangle}{5}$$

$$\frac{d\vec{T}}{dt} = \left\langle -\frac{3}{5}\cos t,\ -\frac{3}{5}\sin t,\ 0 \right\rangle$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\left(-\frac{3}{5}\cos t\right)^2 + \left(-\frac{3}{5}\sin t\right)^2 + 0} = \sqrt{\left(\frac{3}{5}\right)^2} = \frac{3}{5}$$

$$\text{So } \vec{N}(t) = \frac{\left(\frac{d\vec{T}}{dt}\right)}{\left|\frac{d\vec{T}}{dt}\right|} = \langle -\cos t,\ -\sin t,\ 0 \rangle$$



Remark: If we changed 3 and 4 to other numbers, $\vec{N}$ is the same, always parallel to to the xy-plane & points inward toward the z-axis.