14.4 Length of curves

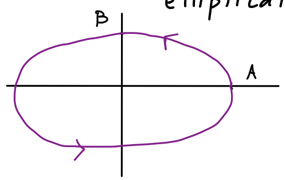
Def (Sec 14.3) If $\vec{r}(t)$ is position, the velocity is the vector fun $\vec{v}(t) = \vec{r}\,'(t)$

speed is scalar fun $|\vec{v}(t)|$

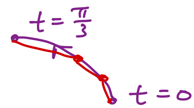
(real-valued)

"Kepler's first law" how a planet revolves about the sun $A > B > 0$

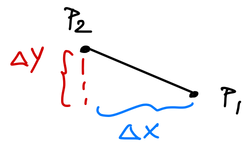
Ex 1: Let C be the elliptical curve $\vec{r}(t) = \langle A\cos t,\ B\sin t \rangle$, $0 \leq t \leq 2\pi$



Consider the segment C_1 from $t=0$ to $t = \frac{\pi}{3}$



- Take n points on this segment, & sum up the distances between adjacent points.
- The limit of this sum as $n \to \infty$ is the (arc) length of C_1
- Distance between two points P_1, P_2 is $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ (by Pythagorean Thm)



- Sum is $\sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$, take limit as $n \to \infty$

Def Arc length of the 2D curve $\vec{r}(t) = \langle f(t), g(t) \rangle$ traversed once for $a \leq t \leq b$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt$$

Arc length of the 3D curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ on the interval $[a, b]$ is

$$L = \int_a^b \underbrace{\sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}}_{\text{magnitude of the derivative}}\, dt = \int_a^b |\vec{r}\,'(t)|\, dt = \int_a^b |\vec{v}(t)|\, dt$$

<u>Back to Ex 1</u>: Length of the elliptical orbit is

$$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-a \sin t)^2 + (b \cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt$$

If $a=b$, then $L = \int_0^{2\pi} a = 2\pi a$

If $a \neq b$, the integrand has no elementary antiderivative,

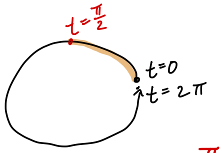
but we can approximate the value using numerical methods.

<u>Def</u> (the natural parameterization: "Arc-length parameterization")

A curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is said to be <u>parameterized by arc length</u> if the parameter t corresponds to arc length.

Ex: $\langle \cos t, \sin t \rangle$, for $0 \leq t \leq 2\pi$ and $\langle \cos 2t, \sin 2t \rangle$, for $0 \leq t \leq \pi$

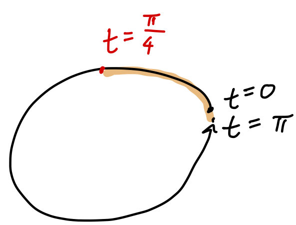
both describe the unit circle centered at $(0,0)$, however...



Length from $t=0$ to $\frac{\pi}{2}$ is $L = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

This is the most natural parameterization.

Intuitively, t increases at the same speed that length of curve increases.



Length from $t=0$ to $\frac{\pi}{4}$ is $L = \frac{\pi}{2} \neq \frac{\pi}{4} - 0$

parameter t DOES NOT correspond to length

In general ...

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$ be a smooth curve.

The (arc) length from $\vec{r}(a)$ to $\vec{r}(t)$ is

$$s(t) := \int_a^t \sqrt{f'(u)^2 + g'(u)^2 + h'(u)^2}\, du = \int_a^t |\vec{r}'(u)|\, du = \int_a^t |\vec{v}(u)|\, du$$

because $\vec{r}' = \vec{v}$ (def earlier)

Def $s(t)$ is called the arc length function of $\vec{r}(t)$.

- Then $\frac{dL}{dt} = |\vec{v}(u)|$ by FTC (Fundamental Thm of Calculus) part I $\left(\frac{d}{dt} \int_a^t \text{(scalar) function}(x)\, dx = \text{(scalar) function}(t) \right)$

- If $|\vec{v}(u)| = 1$ always, then $s(t) = \int_a^t |v(u)|\, du = \int_a^t 1\, du = t - a$

  and so the parameter t corresponds to length $s(t)$.

Thm If $|\vec{r}'(t)| = 1$ (that is, $|\vec{v}(t)| = 1$) for all t in $[a, b]$,

then "the curve is parametrized by its arc length"

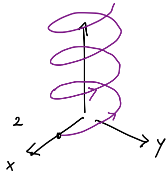
or "the curve uses arc length as a parameter"

Ex 3 (MML #6-8)

Consider the curve (helix) $\vec{r}(t) = \langle 2\cos t, 2\sin t, 4t \rangle$ for $0 \le t \le 999$

$a = 0$ b

projection on the xy-plane is $\langle 2\cos t, 2\sin t \rangle$, circle (counterclock) w/ radius 2, z component increaseas as t increases



a) Compute the arc length function $s(t)$ of $\vec{r}(t)$:

$$\vec{v}(t) = \vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 4 \rangle$$

$$|\vec{v}(t)| = \sqrt{\underbrace{4\sin^2 t + 4\cos^2 t}_{4} + 4^2} = \sqrt{4+4^2} = \sqrt{20} = 2\sqrt{5}$$

$$s(t) \overset{\text{def}}{=} \int_0^t |\vec{v}(u)|\,du = \int_0^t 2\sqrt{5}\,du = 2\sqrt{5}\Big|_{u=0}^{u=t} = 2\sqrt{5}(t-0)$$

$$= \boxed{2\sqrt{5}\,t}$$

our a is 0

b.) Q: Does this curve use arc length as a parameter?

Ans: No because $|\vec{v}(t)| = 2\sqrt{5} \ne 1.$

Find the description that uses arc length as a parameter:

Sol: Use the original $\bar{r}$ but replace t with $\frac{s}{|v(t)|} = \frac{s}{2\sqrt{5}}$

$$\boxed{\left\langle 2\cos\left(\frac{s}{2\sqrt{5}}\right), 2\sin\left(\frac{s}{2\sqrt{5}}\right), 4\left(\frac{s}{2\sqrt{5}}\right) \right\rangle}$$ for

$$\underset{a}{0} \le \frac{s}{2\sqrt{5}} \le \underset{b}{999} \quad \text{or} \quad \boxed{0 \le s \le 999(2\sqrt{5})}$$

t

□

Extra Ex: (MML #11)

Find length of $\vec{r}(t) = \left\langle \frac{t^2}{2}, \frac{(2t+1)^{\frac{3}{2}}}{3} \right\rangle, 0 \le t \le 18.$

Sol: $\vec{r}\,'(t) = \left\langle \frac{2t}{2}, \frac{\frac{3}{2}(2t+1)^{\frac{3}{2}-1}}{3}(2) \right\rangle$ because derivative of inside $(2t+1)$ is 2

$= \left\langle t, (2t+1)^{\frac{1}{2}} \right\rangle$

$|\vec{r}\,'(t)| = \sqrt{t^2 + 2t+1}$

$= \sqrt{(t+1)^2}$

$= t+1$

$$L = \int_0^{18} |\vec{r}\,'(t)|\, dt = \int_0^{18} t+1 \, dt = \frac{t^2}{2} + t \Big|_0^{18} = \frac{18^2}{2} + 18$$

$= 18\left[\frac{18}{2} + 1\right]$

$= 18\,[10]$

$= \boxed{180}$