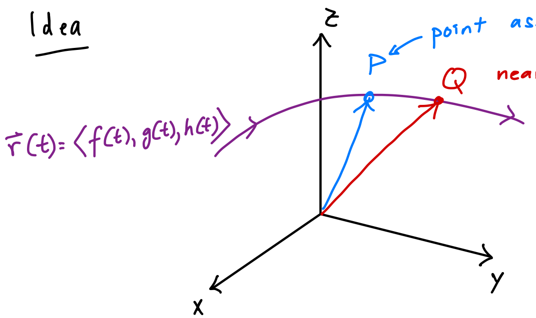
Reading HW for next class:

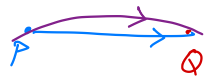
Sec 14.4 : Read the def of arc length
for a vector function

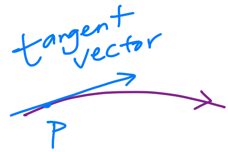
Read Example 2, Example 3

14.2 Calculus of vector-valued functions

Idea







Then $\overrightarrow{PQ}$ is $\vec{r}(t+h) - \vec{r}(t)$

$$\frac{1}{h}\overrightarrow{PQ} = \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$ is in the same direction as $\overrightarrow{PQ}$

As $h \to 0$, Q approaches P, $\dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$ approaches

a vector, denoted $\vec{r}\,'(t)$. This vector $\vec{r}\,'(t)$ is called ...

- a tangent vector at P (if $\vec{r}\,'(t)$ is not the zero vector)
  "because it points in the direction of the curve at P"

- the derivative of $\vec{r}$ with respect to t.

  * If $\vec{r}(t)$ is the position function of a moving object, then $\vec{r}\,'(t)$ is the velocity vector of the object, which always points in the direction of motion, and $|\vec{r}\,'(t)|$ is the speed of the object.

Def If $\vec{r} = \langle f(t), g(t), h(t) \rangle$ for $a \leq t \leq b$,

and each is a differentiable function

on the interval (a, b), and

then $\vec{r}$ is differentiable

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$

* If $\vec{r}\,'(t) \neq \vec{0}$, $\vec{r}\,'(t)$ is a tangent vector at the point corresp. to $\vec{r}(t)$.

$\downarrow$

* $\vec{r}(t)$ is smooth on an interval if it's differentiable and $\vec{r}\,'(t) \neq \vec{0}$ on that interval.

* If $\vec{r}(t)$ is smooth, then the unit tangent vector (for a particular value of t) is

$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$ "divide by its magnitude"

Ex 2 (a) $\vec{r}(t) = \langle t^2, 4t, 4\ln t \rangle$ for $t > 0$ (t in $(\underset{a}{0}, \underset{b}{\infty})$)

(i) Derivative of $\vec{r}(t)$ is

$$\boxed{\vec{r}\,'(t) = \left\langle 2t, 4, \frac{4}{t} \right\rangle}$$

(ii) Unit tangent vector ?

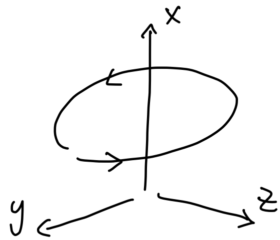
Magnitude of $\vec{r}\,'(t)$ is $|\vec{r}\,'(t)| = \sqrt{(2t)^2 + 4^2 + \left(\frac{4}{t}\right)^2}$

$$= \sqrt{4t^2 + 16 + \frac{16}{t^2}} = \sqrt{4\left(t^2 + 4 + \frac{4}{t^2}\right)} = \sqrt{4\left(t^2 + 2t\frac{2}{t} + \left(\frac{2}{t}\right)^2\right)}$$

$$= \sqrt{4\left(t + \frac{2}{t}\right)^2} = 2\left(t + \frac{2}{t}\right) = 2t + \frac{4}{t}$$

So, $$\boxed{\vec{T}(t) = \frac{1}{2t + \left(\frac{4}{t}\right)} \left\langle 2t, 4, \frac{4}{t} \right\rangle}$$

Ex 2(b): $\vec{r}(t) = 10\,\hat{i} + 3\cos t\,\hat{j} + 3\sin t\,\hat{k}$ for $0 \le t \le 2\pi$



circle at height 10,
radius 3

(i) $\vec{r}\,'(t) = \langle 0, -3\sin t, 3\cos t \rangle$

(ii) $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{1}{3}\langle 0, -3\sin t, 3\cos t\rangle = \boxed{\langle 0, -\sin t, \cos t\rangle}$

$$|\vec{r}\,'(t)| = \sqrt{0^2 + 3^2\sin^2 t + 3^2\cos^2 t}$$

$$= \sqrt{0 + 3^2} \qquad \text{because} \quad \sin^2\theta + \cos^2\theta = 1$$

$$= 3$$

Def Let $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ be a vector function.

An antiderivative of $\vec{r}(t)$ is a vector function $\vec{R}(t)$ s.t

$\vec{R}\,'(t) = \vec{r}(t)$, that is,

$\vec{R}(t) = \langle F(t), G(t), H(t)\rangle$ s.t F is an antiderivative of f,
G — " — of g,
H — " — of h.

The indefinite integral of $\vec{r}$, denoted $\int \vec{r}(t)\,dt$,

is the collection of all antiderivatives of $\vec{r}$.

<u>Fact</u> If $\vec{R}(t)$ is an antiderivative of $\vec{r}$, then the indefinite integral of $\vec{r}$ is

$$\int \vec{r}(t)\, dt = \vec{R}(t) + \vec{C}$$

where $\vec{C} = \langle C_1, C_2, C_3 \rangle$ is an arbitrary constant vector.

<u>Ex 6</u> (i) Find the indefinite integral of $\langle 10, \sin t, t \rangle$

Sol: $\int \langle 10, \sin t, t \rangle\, dt = \langle 10t, -\cos t, \frac{t^2}{2} \rangle + \underbrace{\langle C_1, C_2, C_3 \rangle}_{\vec{C}}$

(Like MML # 11)

(ii) Find the function $\vec{r}(t)$ that satisfies the following conditions. $\vec{r}\,'(t) = \langle 10, \sin t, t \rangle$ and $\vec{r}(0) = \hat{j}$

Sol: $\vec{r}(t) = \langle 10t, -\cos t, \frac{t^2}{2} \rangle + \langle C_1, C_2, C_3 \rangle$ from above.

Set $\vec{r}(0) = \hat{j} = \langle 0, 1, 0 \rangle$

$$\langle 10(0), -\cos(0), \tfrac{0}{2} \rangle + \langle C_1, C_2, C_3 \rangle = \langle 0, 1, 0 \rangle$$

$$\langle 0, -1, 0 \rangle + \langle C_1, C_2, C_3 \rangle = \langle 0, 1, 0 \rangle$$

$$0 + C_1 = 0, \quad -1 + C_2 = 1, \quad 0 + C_3 = 0$$

$$C_1 = 0 \qquad C_2 = 2, \qquad C_3 = 0$$

Ans: $\boxed{\vec{r}(t) = \langle 10t, -\cos t + 2, \frac{t^2}{2} \rangle}$

<u>Def</u> The definite integral of $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

on the interval $[a,b]$ (if each of f, g, h is integrable on $[a,b]$)

is $$\int_a^b \vec{r}(t)\, dt = \left\langle \int_a^b f(t)\, dt, \int_a^b g(t)\, dt, \int_a^b h(t)\, dt \right\rangle$$

MML 13: Evaluate $\displaystyle\int_0^2 t e^t \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) dt$

Int by Parts: $\int u\, dv + \int v\, du = uv$

$\Rightarrow \boxed{\int u\, dv = uv - \int v\, du}$

$$\int_0^2 t e^t\, dt = t e^t \Big|_0^2 - \int_0^2 e^t\, dt = \Big[t e^t - e^t\Big]_0^2$$

$$= 2e^2 - e^2 - (0 - e^0)$$

$$= e^2 + 1$$

$u = t$	$dv = e^t\, dt$
$du = dt$	$v = e^t$

Ans: $\boxed{\left\langle (e^2+1)\,4,\ (e^2+1)\,5,\ (e^2+1)\,6 \right\rangle}$

Additional Examples

EX 3(a)

Let $\vec{v}(t) = \sin t\, \hat{\imath} + 2\cos t\, \hat{\jmath} + \cos t\, \hat{k}$

Compute $\frac{d}{dt}\left(\vec{v}(t^2)\right)$

Sol:

$$\vec{v}\,'(t) = \underbrace{\langle \cos t,\ -2\sin t,\ -\sin t \rangle}_{\downarrow \text{ sub } t^2}$$

$$\frac{d}{dt}\left(\vec{v}(t^2)\right) = \vec{v}\,'(t^2)\,\frac{d}{dt}(t^2) = \langle \cos(t^2), -2\sin(t^2), -\sin(t^2) \rangle\ 2t$$

$$= 2t\cos(t^2)\,\hat{\imath} - 4t\sin(t^2)\,\hat{\jmath} - 2t\sin(t^2)\,\hat{k}$$

Ex 5

$$\int \left(\frac{t}{\sqrt{t^2+2}}\,\hat{\imath} + e^{-3t}\,\hat{\jmath} + (\sin 4t + 1)\,\hat{k} \right) dt$$

$u = t^2 + 2$

$du = 2t\, dt \Rightarrow \frac{1}{2}\, du = t\, dt$

$$\int \frac{t}{\sqrt{t^2+2}}\, dt = \int \frac{1}{\sqrt{u}}\, du = \int u^{-\frac{1}{2}}\, du = \frac{u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + C_1 = 2\sqrt{t^2+2} + C_1$$

$$= \left\langle 2\sqrt{t^2+2} + C_1,\ -\frac{1}{3}e^{-3t} + C_2,\ \frac{-\cos(4t)}{4} + t + C_3 \right\rangle$$