News

Reading HW to do prior to next class:

Textbook Sec 13.6

Example 1 (cylinders),

Example 2 (ellipsoid),

Example 3 (elliptic paraboloid)

What are the basic geometrical objects in 3D?

How do we describe them w/ equations?

13.5 Lines & planes in space

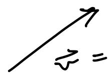
I. Lines in 3D

- Two pts in $\mathbb{R}^3$ determine a unique line

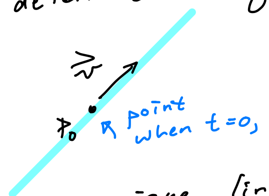
  (just like how two points in $\mathbb{R}^2$ determine a unique line)

- In $\mathbb{R}^3$, one point and a direction (vector) determine a unique line ℓ

  $P_0(x_0, y_0, z_0)$



$\vec{v} = \langle a, b, c\rangle$



(in $\mathbb{R}^2$, one point & a slope determine a unique line)

- The vector equation for the line ℓ, passing through $P_0(x_0, y_0, z_0)$ in the direction of the vector $\vec{v} = \langle a, b, c\rangle$ is

$$\underbrace{\langle x, y, z\rangle}_{\substack{\text{The position vector}\\ \text{from origin to } P(x,y,z),\\ \text{a point on } \ell}} = \underbrace{\langle x_0, y_0, z_0\rangle}_{\substack{\text{position vector}\\ \text{from origin}\\ \text{to } P_0(x_0,y_0,z_0)}} + t\,\underbrace{\langle a, b, c\rangle}_{\vec{v}} \quad \text{for real numbers } t$$

or write $\vec{r} = \vec{r_0} + t\vec{v}$ for short

- The parametric equations for the same line ℓ is

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\} \quad \text{for } -\infty < t < \infty$$

Ex: Ex 2 (book)

Let ℓ be the line through $(-3,5,8)$ and $(4,2,-1)$.

a. Find the vector equation for ℓ

b. Find parametric equations of the line segment from $(-3,5,8)$ to $(4,2,-1)$

Let P_0 $\quad$ P_1

Sol:

a) Direction of the line is $\vec{v} := \overrightarrow{P_0 P_1} = \langle 4-(-3), 2-5, -1-8 \rangle$
$= \langle 7, -3, -9 \rangle$

Take $\vec{r_0} := \langle -3, 5, 8 \rangle$ (We can also take $\vec{r_0} := \langle 4, 2, -1 \rangle$)
(from P_1)

components of P_0

this operation is scalar multip.

A vector equation is $\underbrace{\langle x, y, z \rangle}_{\vec{r}} = \underbrace{\langle -3, 5, 8 \rangle}_{\vec{r_0}} + t \underbrace{\langle 7, -3, -9 \rangle}_{\vec{v}}$

$\vec{r} = \vec{r_0} + t\,\vec{v}$

or $\langle x, y, z \rangle = \langle -3, 5, 8 \rangle + \langle 7t, -3t, -9t \rangle$

or $\langle x, y, z \rangle = \langle -3+7t, 5-3t, 8-9t \rangle$

set each component equal

b) Parametric equations for ℓ are

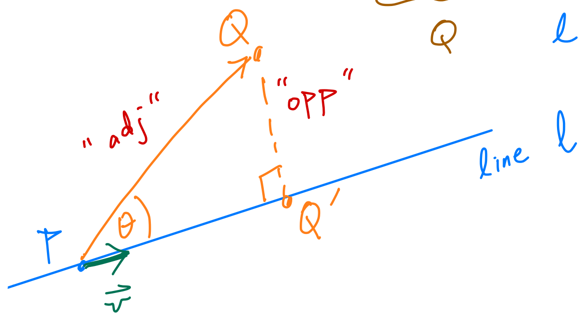
$$\left.\boxed{\begin{aligned} x &= -3+7t \\ y &= 5-3t \\ z &= 8-9t \end{aligned}}\right\} \text{ for } -\infty < t < \infty$$

To find correct interval for t for the line segment,
note when $t=0$, $(x,y,z) = (-3,5,8) = P_0$.
Set $(x,y,z) = P_1 = (4,2,-1) \Rightarrow 4 = -3+7t \Rightarrow 7 = 7t \Rightarrow t = 1$

Interval of t should be: $\boxed{0 \le t \le 1}$

II. "shortest" Distance between a point & a line

point: Q line: ℓ, containing a pt P



The distance between pt Q and line ℓ is the distance d between Q and Q':

$$\sin\theta = \frac{\text{opp}}{\text{adj}} \Rightarrow d = \text{"opp"} = \text{adj}\ \sin\theta = |\vec{PQ}|\sin\theta$$

So $d = |\vec{PQ}|\sin\theta \dfrac{|\vec{v}|}{|\vec{v}|} = \dfrac{|\vec{v}\times\vec{PQ}|}{|\vec{v}|}$ def from 13.4

So the distance between point Q and line ℓ can be given by the formula

$$d = \frac{|\vec{v}\times\vec{PQ}|}{|\vec{v}|}$$

where P is a point in line ℓ and $\vec{v}$ is a vector parallel to ℓ.

Note: In this formula, <u>ANY</u> point P in ℓ and <u>ANY</u> vector $\vec{v}$ parallel to ℓ can be used to get the same answer.

(MML #6)

Ex:

What is the distance between the point $Q(3,4,1)$ and the line $x=4-4t$, $y=3-t$, $z=-3+t$?

Sol:

- We can use the formula from the previous page.
- Choose $P=(4,3,-3)$ and $\vec{v}=\langle -4,-1,1\rangle$ for convenience.

$$d = \frac{\left|\underbrace{\langle -4,-1,1\rangle}_{\vec{v}} \times \underbrace{\langle 3-4,\ 4-3,\ 1-(-3)\rangle}_{\overrightarrow{PQ}}\right|}{|\langle -4,-1,1\rangle|} = \cdots$$

$$= \frac{|-5\hat{i} + 15\hat{j} - 5\hat{k}|}{\sqrt{16+1+1}}$$

$$= \frac{\sqrt{25+225+25}}{\sqrt{18}}$$

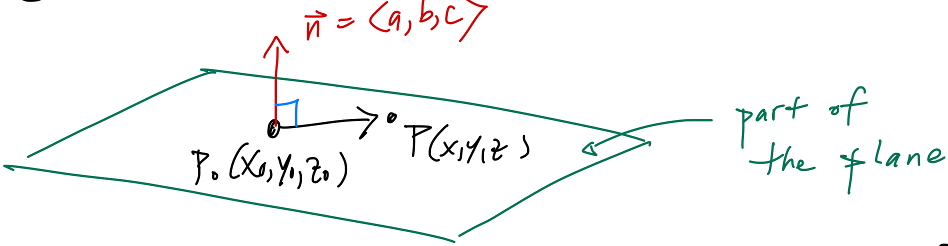
$$= \frac{\sqrt{275}}{\sqrt{18}}$$

$$= \frac{5\sqrt{11}}{3\sqrt{2}}$$

$$= \frac{5\sqrt{22}}{6} \quad \square$$

III. Planes

A plane (flat surface w/ infinite extent in all directions) is uniquely determined by a point P_0 and a normal vector $\vec{n}$:



Every point $P(x, y, z)$ on this plane satisfies...

- $\overrightarrow{P_0P}$ lies in the plane
- the vector $\overrightarrow{P_0P}$ is orthogonal to $\vec{n}$,

  meaning $\vec{n} \cdot \overrightarrow{P_0P} = 0$,

  i.e. $\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

So, $$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

is an equation of the plane passing through point $P_0(x_0, y_0, z_0)$ & w/ a normal vector $\vec{n} = \langle a, b, c \rangle$.

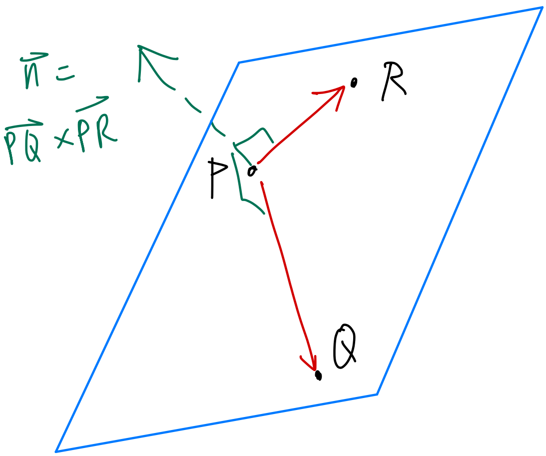
Equivalently, write $ax + by + cz = k$

where $k = ax_0 + by_0 + cz_0$.

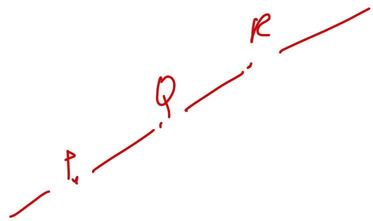
Ex: Ex 6 (book)

Find an equation of the plane that passes through the points $P(2,-1,3)$, $Q(1,4,0)$, and $R(0,-1,5)$.

Sol:



Make sure the pts aren't all on the same line:



- Set $\vec{n} := \vec{PQ} \times \vec{PR}$ (can also do $\vec{QR} \times \vec{QP}$, etc)

$$\vec{n} = \langle 1-2, 4-(-1), 0-3 \rangle \times \langle 0-2, -1-(-1), 5-3 \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 5 & -3 \\ -2 & 0 & 2 \end{vmatrix} = \cdots = 10\hat{i} + 8\hat{j} + 10\hat{k}$$

- Choose $P_0 := P$ (can also choose $P_0 := Q$ or R)
- Then an equation of the plane is

$$\boxed{10(x-2) + 8(y-(-1)) + 10(z-3) = 0}$$

- If we choose $P_0 = Q$, we would get

$$10(x-1) + 8(y-4) + 10z = 0. \quad \square$$

IV Planes that are parallel vs orthogonal (vs neither)

Def: Two planes are parallel if their normal vectors are parallel (they are scalar multiples of each other).

- Two planes are orthogonal if their normal vectors are orthogonal (their dot product is 0)

Ex 9(book), also MML #13

Ex:

Find an equation of the line where the planes $Q: x+2y+z=5$ and $R: 2x+y-z=7$ intersect.

Sol:

- The normal vectors of the planes Q and R,

$$\vec{n}_Q = \langle 1, 2, 1 \rangle \text{ and } \vec{n}_R = \langle 2, 1, -1 \rangle$$

are not scalar multiples of each other.

- So the planes are not parallel, hence they must intersect in a line ℓ.

(con't to the next page)

• To describe the line ℓ, we need:

1. a point P_0 on ℓ
2. a vector $\vec{v}$ parallel to ℓ

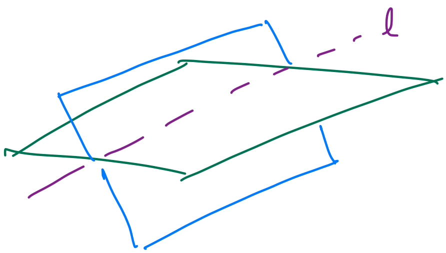
1. Find P_0: * Set $z=0$ to find a point on the xy-plane:

$$\left.\begin{array}{l} *\text{Eq for plane } Q: x+2y=5 \\ *\text{Eq for plane } R: 2x+y=7 \end{array}\right\} \Rightarrow \begin{array}{r} x+2y=5 \\ -4x-2y=-14 \;+ \\ \hline -3x \quad\;\; =-9 \end{array} \Rightarrow x=3$$

$$(3)+2y=5 \Rightarrow \boxed{y=1}$$

So, take $P_0(3,1,0)$ we set $z=0$

2. Find $\vec{v}$: * Because ℓ lies in both plane Q and plane R, ℓ is orthogonal to both the normal vectors $\vec{n}_Q$ and $\vec{n}_R$



* Since $\vec{n}_Q \times \vec{n}_R$ is orthogonal to both $\vec{n}_Q$ and $\vec{n}_R$, we know $\vec{n}_Q \times \vec{n}_R$ is parallel to ℓ:

so take $\vec{v} = \vec{n}_Q \times \vec{n}_R = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \langle -3, 3, -3 \rangle$

Therefore, a possible vector equation for the line ℓ is

$$\langle x, y, z\rangle = \overbrace{\langle 3, 1, 0\rangle}^{\overrightarrow{OP_0}} + t\,\overbrace{\langle -3, 3, -3\rangle}^{\vec{v}} \quad \text{for } -\infty < t < \infty$$

and a possible parametric equation for ℓ is

$$x = 3-3t, \quad y = 1+3t, \quad z = -3t \quad \text{for } -\infty < t < \infty.$$

$\square$

Extra Ex (MML # 9)

Find the point at which the plane $-4x + 9y - 2z = 13$ & the line $x = 10 + 9t,\ y = -4 - 6t,\ z = 7 - 6t$ intersect

<u>Sol:</u> Sub the line parametric eqs into plane eq:

$$-4(10 + 9t) + 9(-4 - 6t) - 2(7 - 6t) = 13$$

$$-40 - 36t - 36 - 54t - 14 + 12t = 13$$

$$-90 - 78t = 13$$

$$-103 = 78t$$

$$\boxed{-\frac{103}{78} = t}$$

sub into parametric eqs of line to find x, y, z

$$x = 10 + 9\left(-\frac{103}{78}\right)$$

$$y = -4 - 6\left(-\frac{103}{78}\right)$$

$$z = 7 - 6\left(-\frac{103}{78}\right)$$

Ans: the point $\left(-\frac{49}{26}, \frac{51}{13}, \frac{194}{13}\right)$.