News

- Before next class, read ...

  Textbook Examples 1, 2, 3

  of Sec 12.1 (Parametric equations)

13.4 Cross products

<u>Idea of cross product</u> Given vectors $\vec{a}$ & $\vec{b}$, find a third nonzero vector $\vec{c}$ that is perpendicular to both $\vec{a}$ and $\vec{b}$

Geometric def of cross product

If θ is the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ (so $0 \leqslant \theta \leqslant \pi$), then

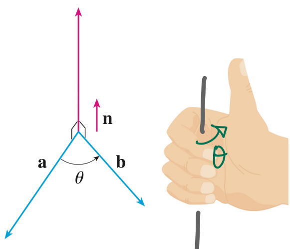
$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \underbrace{|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}| \sin\theta}_{\text{scalar}} \; \underset{\text{unit vector}}{\vec{n}}$$

where $\vec{n}$ is ...

(i) a unit vector (meaning $|\vec{n}| = 1$)

(ii) orthogonal (or normal or perpendicular) to both $\vec{a}$ and $\vec{b}$

(iii) the direction of $\vec{n}$ is given by the right-hand rule



Q: What is the length of $\vec{a} \times \vec{b}$? Hint: $\vec{n}$ is unit vector

Ans: $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$

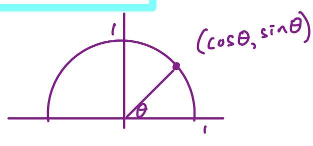
Fact

Two nonzero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are parallel if and only if

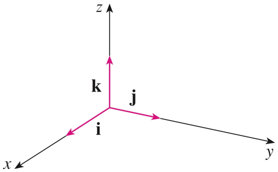
$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$$

Why?

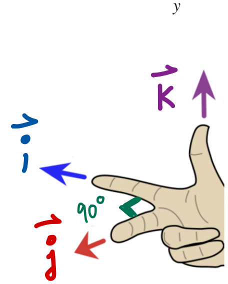
$\sin\theta$ is 0 if and only if $\theta = 0$ or π



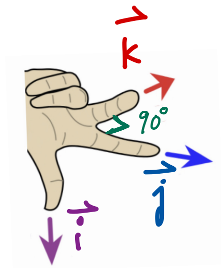
Recall the coordinate unit vectors (standard basis vectors)



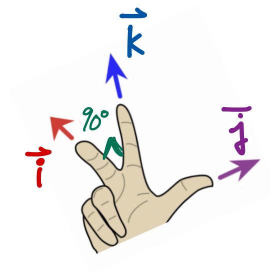
$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$



$\mathbf{i} \times \mathbf{j} = \mathbf{k}$

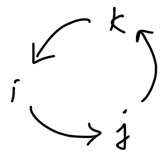


$\mathbf{j} \times \mathbf{k} = \mathbf{i}$



$\mathbf{k} \times \mathbf{i} = \mathbf{j}$

"To help you remember":



$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

Some facts about the cross product

- $\boxed{\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \quad \text{for all vectors } \vec{a}, \vec{b}}$
- In general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

Not associative!

E.g.

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

Algebraic Def cross products

A **determinant of order 2** is defined by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

E.g. $\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} = 1(7) - 3(2) = 1$

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Write the det of a 3x3 matrix $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ "just a symbol to help us memorize the def"

called a 3x3 matrix

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Def

$$\boxed{\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}}$$

Example:

If $\mathbf{a} = \langle 1, 3, 4\rangle$ and $\mathbf{b} = \langle 2, 7, -5\rangle$,

find a unit vector with positive first coordinate orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Solution

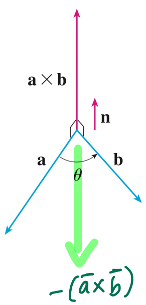
Note: Any scalar multiple of $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ & $\vec{b}$

(i) If $\mathbf{a} = \langle 1, 3, 4\rangle$ and $\mathbf{b} = \langle 2, 7, -5\rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$= (-15 - 28)\,\mathbf{i} - (-5 - 8)\,\mathbf{j} + (7 - 6)\,\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$



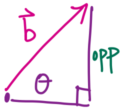
(ii) To get a vector in opposite direction as $\vec{a} \times \vec{b}$, do scalar multiplication

$$-1\left(-43\mathbf{i} + 13\mathbf{j} + \mathbf{k}\right) = \langle 43, -13, -1\rangle$$

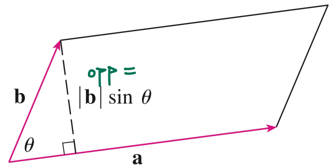
(iii) To get a unit vector, scalar multiply

$$\frac{1}{\ell}\langle 43, -13, -1\rangle \quad \text{where } \ell = \underbrace{\sqrt{(43)^2 + (-13)^2 + (-1)^2}}_{\text{length of } \vec{a} \times \vec{b}}$$

Physical meaning of $|\vec{a} \times \vec{b}|$



$\sin\theta = \dfrac{\text{opp}}{|\vec{b}|} \Rightarrow$



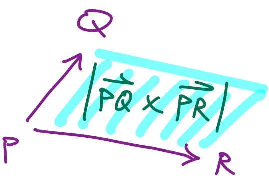
Area of parallelogram is (base) (height)

$= |\vec{a}| \quad |\vec{b}| \sin\theta$

Since $|\vec{a} \times \vec{b}| = \underbrace{|\vec{a}|\,|\vec{b}|\,\sin\theta}_{\text{length of } \vec{a}\times\vec{b}}$, we have ...

> The length of the cross product **a** × **b** is equal to the area of the parallelogram determined by **a** and **b**.

EXAMPLE Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.



$$\overrightarrow{PQ} = (-2 - 1)\,\mathbf{i} + (5 - 4)\,\mathbf{j} + (-1 - 6)\,\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{PR} = (1 - 1)\,\mathbf{i} + (-1 - 4)\,\mathbf{j} + (1 - 6)\,\mathbf{k} = -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5 - 35)\,\mathbf{i} - (15 - 0)\,\mathbf{j} + (15 - 0)\,\mathbf{k} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$$

Area of parallelogram is $|\overrightarrow{PQ} \times \overrightarrow{PR}| \overset{\text{length formula}}{=} \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$

The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$. ■