News:

- Drop-in office Hours: Wed 2–3:15 pm
  Fri 1:30–3:15 pm

  @ Southwick Hall 350 M

- Also available by appointments on MWF morning & afternoon
  (at Southwick Hall or via Zoom)

- MML 13.1, 13.2 due tonight
  (two more weeks to complete w/ 20% off each late problem)

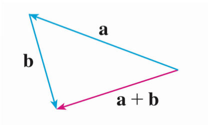
- Before next class, read ...

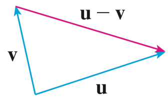
  * Example 1, 2, 4, 3
    (in order of importance)
    from Sec 13.4: Cross products

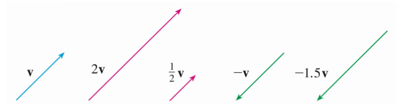
  * "right-hand rule" from Sec 13.2: 3D Vectors

Recall

- Sum of two vectors
- Difference of two vectors
- multiplying a vector by a number / scalar







- Can we "multiply" two vectors ?

dot product : " product" of two vectors is a number

cross product : " product" of two vectors is a vector

13.3 Dot Products (also called the scalar products or inner products)

1 Definition

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

Note $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (The order of the vectors doesn't matter)

EXAMPLE

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\tfrac{1}{2} \rangle = (-1)(6) + 7(2) + 4(-\tfrac{1}{2}) = 6$$

Fact

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$

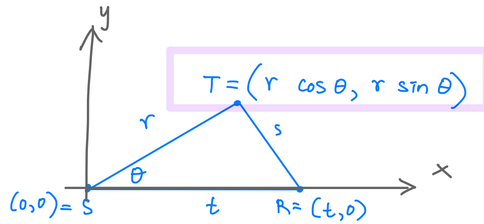
Why ?

$$\vec{a} \cdot \vec{a} \overset{def}{=} \langle a_1, a_2 \rangle \cdot \langle a_1, a_2 \rangle = a_1^2 + a_2^2$$

$$|\vec{a}|^2 = (\text{length of } \vec{a})^2 = \left(\sqrt{a_1^2 + a_2^2}\right)^2 = a_1^2 + a_2^2$$

Justification for "geometric meaning"

Law of Cosines



$S =$ Distance between R and T

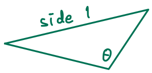
$$= \sqrt{(t - r\cos\theta)^2 + (0 - r\sin\theta)^2}$$

$$s^2 = (t - r\cos\theta)^2 + r^2(\sin\theta)^2$$

$$s^2 = t^2 - 2tr\cos\theta + r^2(\cos\theta)^2 + r^2(\sin\theta)^2$$

$$s^2 = t^2 + r^2 - 2tr\cos\theta \qquad (\text{since } \cos^2\theta + \sin^2\theta = 1)$$

"Law of Cosines"



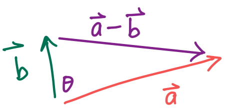
$$(\text{side }1)^2 = (\text{side }2)^2 + (\text{side }3)^2 - 2(\text{side }1)(\text{side }2)\cos\theta$$

Geometric meaning of the dot product

Let $\vec{a} = \langle a_1, a_2 \rangle$

$\vec{b} = \langle b_1, b_2 \rangle$

$\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$



By law of cosines,

① $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

② $|\vec{a} - \vec{b}|^2 \overset{\text{Length formula}}{=} (a_1 - b_1)^2 + (a_2 - b_2)^2$

$$= a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2$$

$$= \underbrace{a_1^2 + a_2^2}_{} + \underbrace{b_1^2 + b_2^2}_{} - 2\underbrace{(a_1b_1 + a_2b_2)}_{\text{dot product}}$$

$$= |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b}$$

① $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

② $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b}$ $\quad -$

$$0 = -2|\vec{a}||\vec{b}|\cos\theta + 2\,\vec{a}\cdot\vec{b}$$

So $\boxed{\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta}$

Geometric meaning of the dot product

Theorem If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

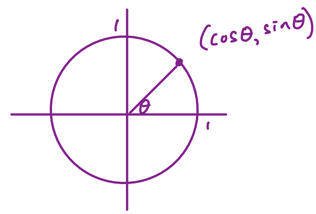
Corollary If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

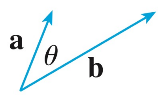
$\cos(\theta)$ is ... positive when $0 \leq \theta < \frac{\pi}{2}$

0 when $\theta = \frac{\pi}{2}$

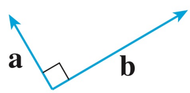
negative when $\frac{\pi}{2} < \theta \leq \pi$



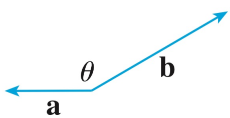
So we can determine acute/right/obtuse angle by computing $\vec{a} \cdot \vec{b}$



$\mathbf{a} \cdot \mathbf{b} > 0$
θ acute



$\mathbf{a} \cdot \mathbf{b} = 0$
$\theta = \pi/2$



$\mathbf{a} \cdot \mathbf{b} < 0$
θ obtuse

Def

Two vectors $\vec{a}, \vec{b}$ are <u>orthogonal</u> or <u>perpendicular</u> if $\vec{a} \cdot \vec{b} = 0$

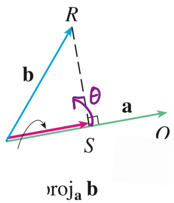
$\vec{a}$ and $\vec{b}$ are parallel ($\theta = 0$ or $\theta = \pi$) iff $\vec{a} \cdot \vec{b} = \pm |\vec{a}| |\vec{b}|$

Ex: $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$

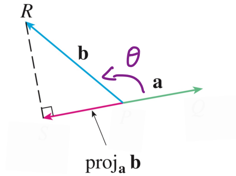
so the angle between $\langle 2, 4 \rangle$ and $\langle 3, -1 \rangle$ is <u>acute</u>

Orthogonal Projections

The (orthogonal) projection of $\vec{b}$ onto $\vec{a}$ is the vector $\text{proj}_{\vec{a}} \vec{b}$



when the angle θ between $\vec{a}$ and $\vec{b}$ is acute (smaller than $\frac{\pi}{2}$)



when the angle θ between $\vec{a}$ and $\vec{b}$ is obtuse (bigger than $\frac{\pi}{2}$)

Think of this vector $\text{proj}_{\vec{a}} \vec{b}$ as a shadow of $\vec{b}$

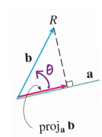
The scalar component of $\vec{b}$ in the direction of $\vec{a}$

(denoted $\text{scal}_{\vec{a}} \vec{b}$)

is the length of $\text{proj}_{\vec{a}} \vec{b}$ if θ is less than $\frac{\pi}{2}$

(-1) times length of $\text{proj}_{\vec{a}} \vec{b}$ if θ is bigger than $\frac{\pi}{2}$

$$\boxed{\text{scal}_{\vec{a}} \vec{b} = |\vec{b}| \cos\theta}$$ why?



$\cos\theta = \frac{\text{adj}}{\text{hyp}}$

Since $\vec{a} \cdot \vec{b} = |a||b| \cos\theta$,

we have $$\boxed{\text{scal}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}}$$

$\text{scal}_{\vec{a}} \vec{b}$ is a number

Vector projection of **b** onto **a**: $\text{proj}_{\mathbf{a}} \mathbf{b} = \underbrace{\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)}_{(\pm 1)\text{ length } \text{comp}_{\vec{a}} \vec{b}} \underbrace{\frac{\mathbf{a}}{|\mathbf{a}|}}_{\text{unit vector in the direction of } \vec{a}} = |\vec{b}| \cos\theta \, \frac{\vec{a}}{|\vec{a}|}$

$\text{proj}_{\vec{a}} \vec{b}$ is a vector

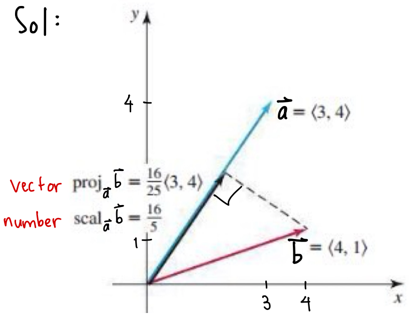
Ex : $\vec{a} = \langle 3,4 \rangle$, $\vec{b} = \langle 4,1 \rangle$

1. Sketch $\underbrace{\text{proj}_{\vec{a}} \vec{b}}$ and $\underbrace{\text{scal}_{\vec{a}} \vec{b}}$

(orthogonal) projection of $\vec{b}$ onto $\vec{a}$

scalar component of $\vec{b}$ in the direction of $\vec{a}$

Sol:



Note: $\text{proj}_{\vec{a}} \vec{b}$ and $\vec{a}$ point in the same direction, so $\text{scal}_{\vec{a}} \vec{b}$ should be a positive number

2. Compute $\text{scal}_{\vec{a}} \vec{b}$

Sol: $\text{scal}_{\vec{a}} \vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \dfrac{\langle 3,4 \rangle \cdot \langle 4,1 \rangle}{|\langle 3,4 \rangle|} = \dfrac{3(4) + 4(1)}{\sqrt{3^2 + 4^2}} = \dfrac{16}{\sqrt{25}} = \boxed{\dfrac{16}{5}}$

use this formula instead of $|\vec{b}| \cos\theta$ because we don't know the angle between $\vec{a}$ and $\vec{b}$

3. Compute $\text{proj}_{\vec{a}} \vec{b}$

Sol: $\text{proj}_{\vec{a}} \vec{b} = \left(\text{scal}_{\vec{a}} \vec{b}\right) \dfrac{\vec{a}}{\underbrace{|\vec{a}|}_{=5}} = \boxed{\dfrac{16}{5} \left\langle \dfrac{3}{5}, \dfrac{4}{5} \right\rangle}$

Similar to MML 10

Parallel and Normal Forces (An application of projections)

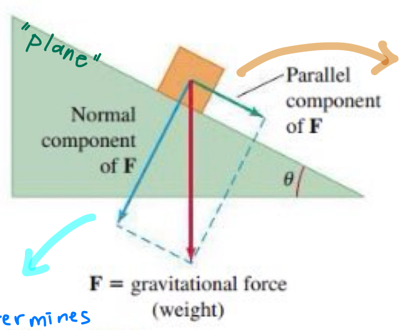
EX: • An object (10 lbs) rests on a plane that is inclined 30° above the horizontal.

• $\vec{F}$ is the (downward) grativational force: $\vec{F} = \langle 0, -10 \rangle$

because object's weight is 10 lbs

• The projection of $\vec{F}$ in the direction parallel to the plane determines the tendency of the object to slide down the plane

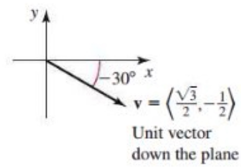
• The projection of $\vec{F}$ in the direction normal (aka perpendicular) to the plane determines the tendency of the object to "stick" to the plane



determines the tendency of the object to slide down the plane

determines tendency of object to "stick" to plane

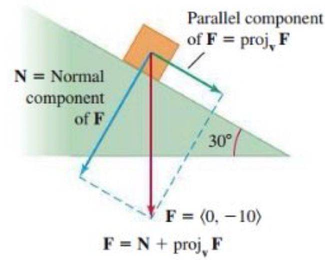
Q: a.) What is the component of the gravitational force $\vec{F}$ parallel to the plane?



Sol: First, find the unit vector $\vec{v}$ w/ direction down the plane:

$$\vec{v} = \langle \cos(-30°), \sin(-30°) \rangle$$

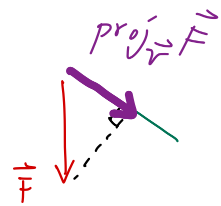
$$= \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$$



Then the component of $\vec{F}$ parallel to the plane is

$$\text{proj}_{\vec{v}} \vec{F} = \left(\frac{\vec{v} \cdot \vec{F}}{|\vec{v}|} \right) \frac{\vec{v}}{|\vec{v}|}$$

equals 1 because $\vec{v}$ is a unit vector



$$= (\vec{v} \cdot \vec{F}) \vec{v}$$

$$= \left(\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle \cdot \langle 0, -10 \rangle \right) \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$$

$$= \boxed{5 \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle}$$

b.) what is the component of the force normal (perdendicular) to the plane? Denote this by $\vec{N}$.

Sol: Note from the picture $\vec{F} = \vec{N} + \text{proj}_{\vec{v}} \vec{F}$

$$\text{so} \quad \vec{N} = \vec{F} - \text{proj}_{\vec{v}} \vec{F} = \langle 0, -10 \rangle - 5 \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$$

$$= \left\langle -5\frac{\sqrt{3}}{2}, -10 + \frac{5}{2} \right\rangle = \boxed{\left\langle -\frac{5\sqrt{3}}{2}, -\frac{15}{2} \right\rangle}$$

Similar to MML 11