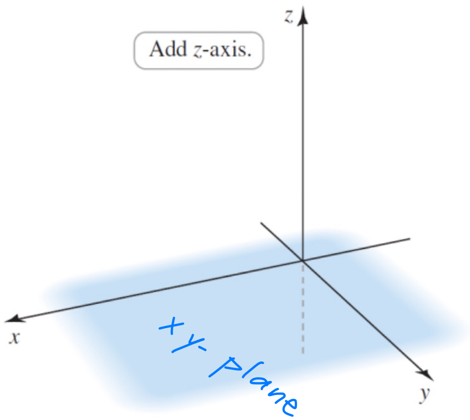
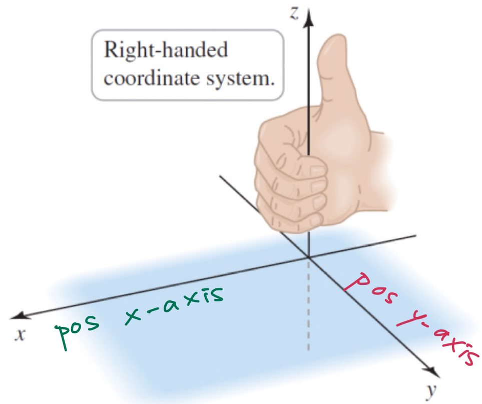
13.2 Vectors in three dimensions

A 3D coordinate system is created by adding the z-axis to the xy-plane

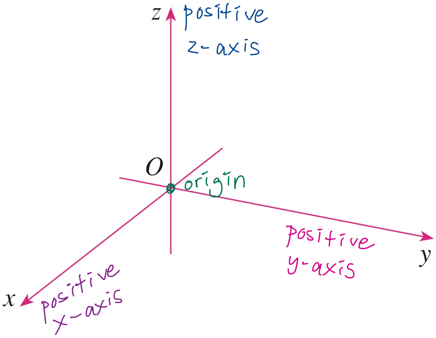




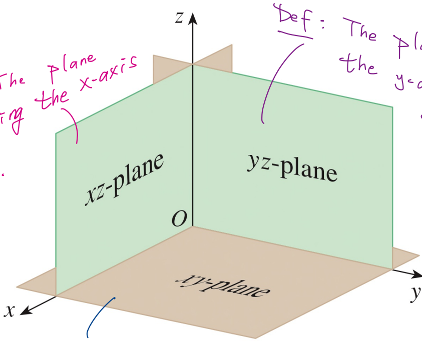
The set of all points described by the triples (x,y,z) is called three-dimensional space, xyz-space, or $\mathbb{R}^3$.

Part I: 3D space

To represent a point in $\mathbb{R}^3$, draw 3 coordinate axes.



Def: The plane containing the x-axis and z-axis.



Def: The plane containing the y-axis and the z-axis.

Def: The xy-plane is the (horizontal) plane containing the x-axis and the y-axis

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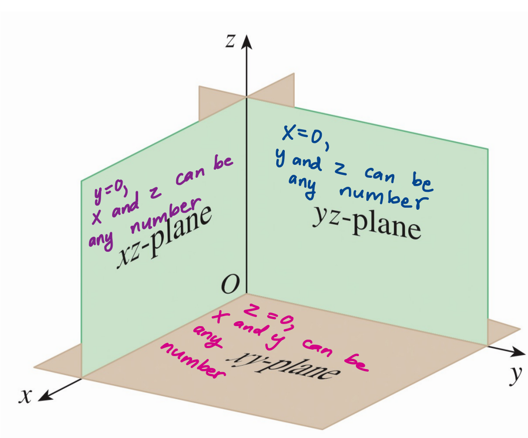
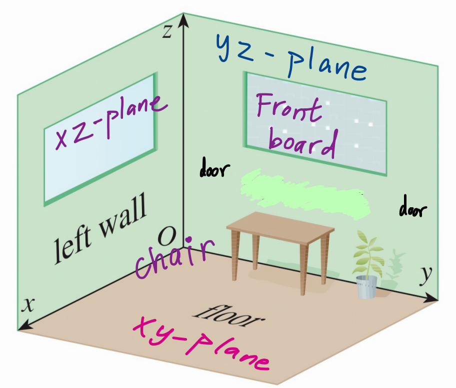
The three planes divide space into eight octants.

The first octant is determined by the positive axes.





Part I: 3D space

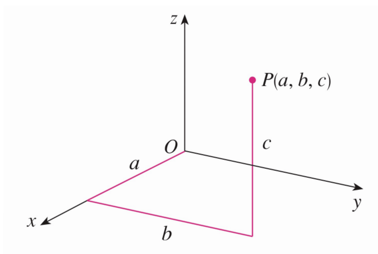
To find the point (a, b, c) in $\mathbb{R}^3$,

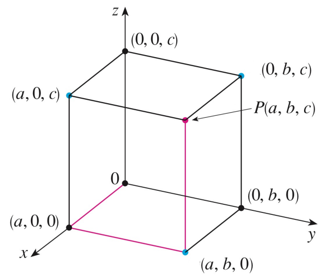
start from the origin O

Move a units along the x-axis

Move b units parallel to the y-axis

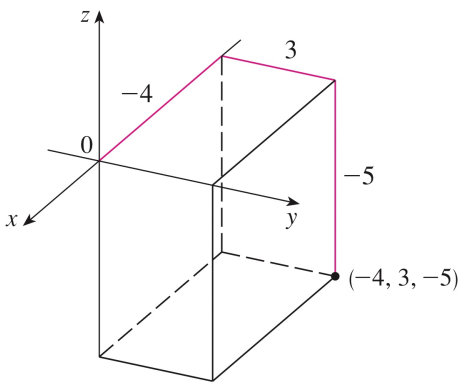
Move c units parallel to the z-axis



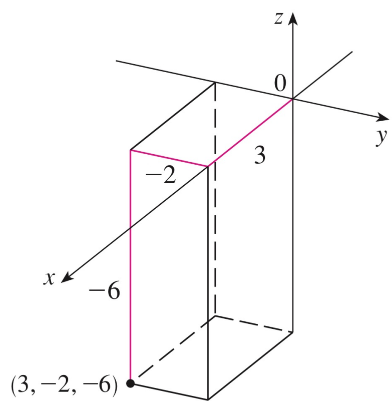


Examples:

point $(-4, 3, -5)$



point $(3, -2, -6)$

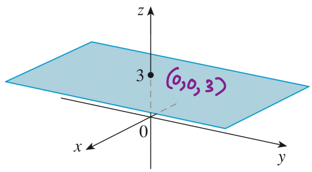


Part II: Equations of simple planes

EXAMPLE 1 What surfaces in $\mathbb{R}^3$ are represented by the following equations?

(a) $z = 3$ (b) $y = 5$

The set of points $(x, y, 3)$ where x and y can be any number

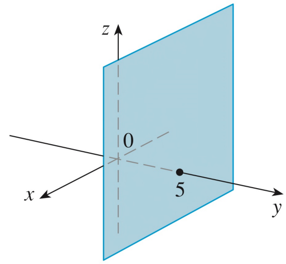


(a) $z = 3$, a plane in $\mathbb{R}^3$

Parallel to the xy-plane $z=0$
(the horizontal "floor")

The set of all points (x,y,z) in 3D space where $y=5$ and x, z can be any number.

Parallel to the xz-plane $y=0$
(the "left wall")



(b) $y = 5$, a plane in $\mathbb{R}^3$

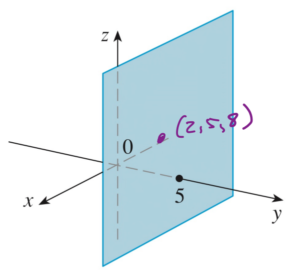
Ex:

y

An equation for the plane parallel to the xz-plane and passing through the point $(2, 5, 8)$ is

Sol: Points on a plane parallel to the xz-plane have the same y-coordinate. Since point $(2,5,8)$ has y-coordinate 5, the equation is $\boxed{y=5}$

(x, z can be any number)



(b) $y = 5$, a plane in $\mathbb{R}^3$

Note:
If we replace "xz-plane" with "yz-plane", the answer would be $\boxed{x=2}$

Part II: Planes

EXAMPLE 3 Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $y = x$.

Go to geogebra.org/3d and type $y=x$

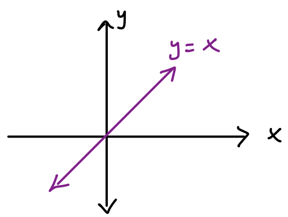
or desmos.com/3d ☑ Extend to 3D

The surface is the set of points $(x,y,z) = (x,x,z)$ where x and z are any numbers.

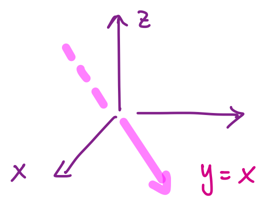
We get a plane containing:

* the z-axis (since $(0,0,z)$ is in the surface for any z)
* the line $y=x$, $z=0$

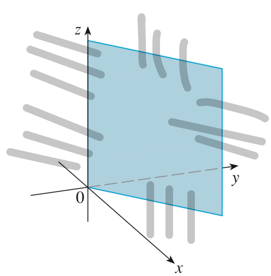
Step 1: Sketch



Step 2: Draw the line from Step 2 "on the floor":

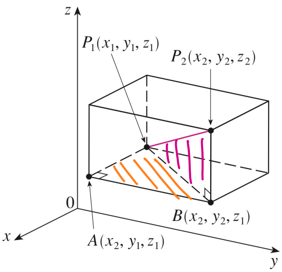


Step 3: Allows any value for z to get an "infinite wall"



Plane $y=x$

Part III: Distance in $\underbrace{xyz\text{-space}}_{\mathbb{R}^3}$



Distance Formula in Three Dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Why?

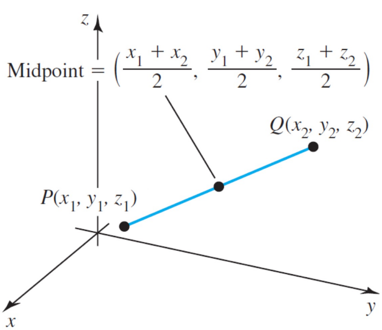
- Apply Pythagorean Theorem to find $|P_1B|$
- Apply it again for

EXAMPLE 4 The distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$ is

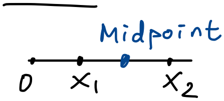
$$|PQ| = \sqrt{(1-2)^2 + (-3+1)^2 + (5-7)^2} = \sqrt{1 + 4 + 4} = 3$$

The midpoint of the line segment of the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\underbrace{\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)}_{\text{averages of the } x\text{-, } y\text{-, and } z\text{-coordinates}}$$



Why?

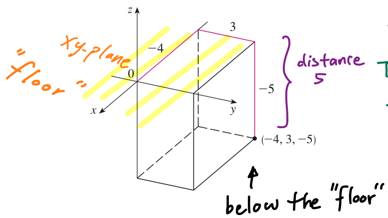


$$\text{Midpoint} = x_1 + \frac{x_2 - x_1}{2}$$

$$= \frac{x_1 + x_2}{2}$$

(Extra Example)

Find the distance from $(-4, 3, -5)$ to the xy-plane.

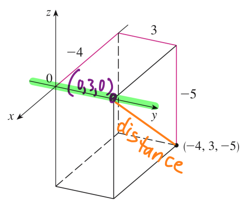


Answer: 5

The xy-plane is the horizontal "floor".

The distance is the difference between $z=0$ and $z=-5$

Find the distance from $(-4, 3, -5)$ to the y-axis.



Answer:

The point on the y-axis closest to $(-4, 3, -5)$ is $(0, 3, 0)$.

Distance : $\sqrt{(-4-0)^2 + (3-3)^2 + (-5-0)^2}$

$= \sqrt{16+25}$

$= \sqrt{41}$

Part IV: Spheres

- A <u>sphere</u> with center (a,b,c) and radius r is the set of all points that are of distance r from the center.

- A <u>ball</u> is the set of all points inside and on the sphere.

The set of points of distance r from (a,b,c) can be described by equation

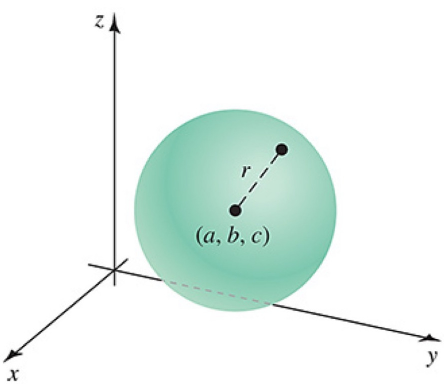
$$r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}, \quad \text{so} \ldots$$

A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

A (closed) **ball** centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x-a)^2 + (y-b)^2 + (z-c)^2 \le r^2.$$



Sphere: $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$
Ball: $(x-a)^2 + (y-b)^2 + (z-c)^2 \le r^2$

These two equations are in <u>standard form</u>

(MML Problem 5)

EXAMPLE 3 **Equation of a sphere** Consider the points $P(1, -2, 5)$ and $Q(3, 4, -6)$. Find an equation of the sphere for which the line segment PQ is a diameter.

SOLUTION The center of the sphere is the midpoint of PQ:

$$\left(\frac{1+3}{2}, \frac{-2+4}{2}, \frac{5-6}{2}\right) = \left(2, 1, -\frac{1}{2}\right).$$

The diameter of the sphere is the distance $|PQ|$, which is

$$\sqrt{(3-1)^2 + (4+2)^2 + (-6-5)^2} = \sqrt{161}.$$

Therefore, the sphere's radius is $\frac{1}{2}\sqrt{161}$, its center is $\left(2, 1, -\frac{1}{2}\right)$, and it is described by the equation

$$(x-2)^2 + (y-1)^2 + \left(z+\frac{1}{2}\right)^2 = \left(\frac{1}{2}\sqrt{161}\right)^2 = \frac{161}{4}.$$

Ex (Identifying equations): Describe the set of points that satisfy the equation

$$x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$$

Answer: $x^2+4x \quad + \quad y^2-6y \quad + \quad z^2+2z \quad = -6$

"Complete the squares" $x^2+4x+2^2 + y^2-6y+3^2 + z^2+2z+1^2 = -6+4+9+1$

$$(x+2)^2 + (y-3)^2 + (z+1)^2 = 8$$

of a sphere

Comparing this equation with the standard form, we see that it is the equation of a sphere with center $\boxed{(-2, 3, -1)}$ and radius $\boxed{\sqrt{8} = 2\sqrt{2}}$.

(MML Problems 6, 7)

EXAMPLE 4 **Identifying equations** Describe the set of points that satisfy the equation $x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$.

SOLUTION We simplify the equation by completing the square and factoring:

$$(x^2 - 2x) + (y^2 + 6y) + (z^2 - 8z) = -1 \quad \text{Group terms.}$$

$$(x^2 - 2x + 1) + (y^2 + 6y + 9) + (z^2 - 8z + 16) = 25 \quad \text{Complete the square.}$$

$$(x-1)^2 + (y+3)^2 + (z-4)^2 = 25. \quad \text{Factor.}$$

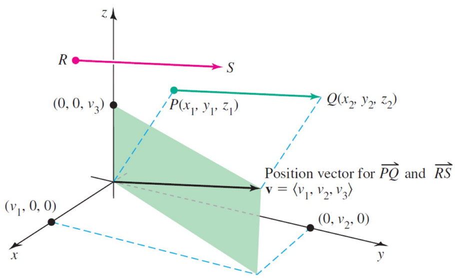
The equation describes a sphere of radius 5 with center $(1, -3, 4)$.

Part V: Vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$ are straightforward extensions of vectors in the xy-plane. We simply add a 3rd component.

- Vectors having the same length and direction are equal.

- The <u>position</u> <u>vector</u> $\vec{v} = \langle v_1, v_2, v_3 \rangle$ has its tail (starting point) at the origin and its head (terminal point) is the point (v_1, v_2, v_3).



Here $\overrightarrow{RS}$ and $\overrightarrow{PQ}$ and $\vec{v}$ are all equal because they have the same magnitude and direction.

<u>Example:</u>

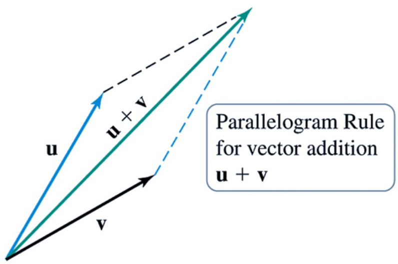
What position vector is equal to the vector from $(-6, 2, 1)$ to $(2, 3, -7)$?

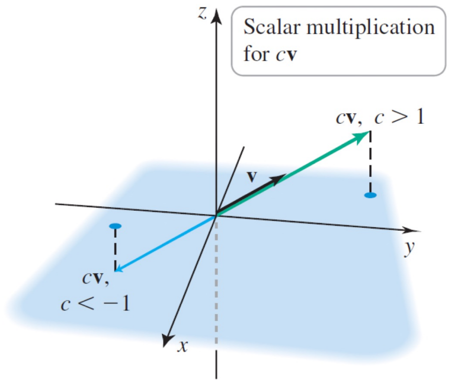
Sol: $\langle 2-(-6), 3-2, -7-1 \rangle = \boxed{\langle 8, 1, -8 \rangle}$

(Extra page)

Sum of two vectors is found geometrically as before

Scalar multiple corresponds to stretching / compressing a vector, with a reversal of direction for negative scalar





Def of vector addition and scalar multiplication:

Let r be a scalar (number), $\vec{a} = \langle a_1, a_2, a_3 \rangle$, & $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

Sum $\quad \vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$.

Scalar multiple $\quad r\vec{a} = \langle ra_1, ra_2, ra_3 \rangle$.

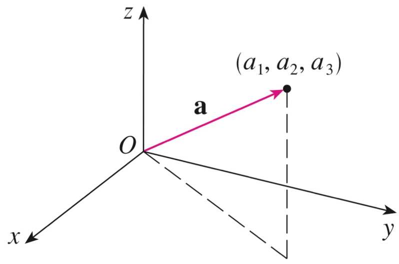
Example:

Let $\mathbf{u} = \langle 2, -4, 1 \rangle$ and $\mathbf{v} = \langle 3, 0, -1 \rangle$

$$\mathbf{u} + 2\mathbf{v} = \langle 2, -4, 1 \rangle + 2\langle 3, 0, -1 \rangle = \langle 8, -4, -1 \rangle$$

Part VI: Magnitude and unit vectors

Again, we simply add a 3rd component



$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

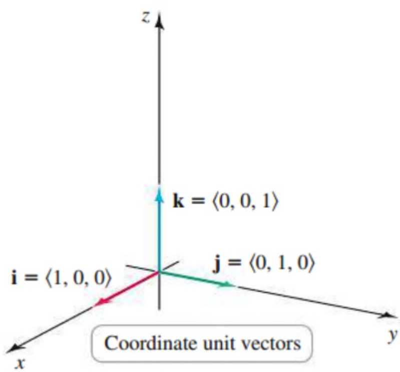
The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

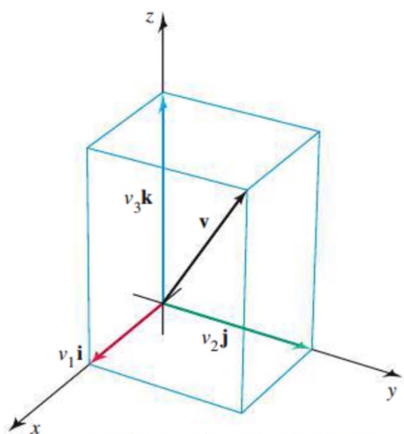
$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Ex: Find the length of the vector $\mathbf{v} = \langle 10, 6, 3 \rangle$. Sol: $\sqrt{10 \cdot 10 + 6 \cdot 6 + 3 \cdot 3} = \sqrt{145}$.

The coordinate unit vectors (or standard basis vectors) in $\mathbb{R}^3$ are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$





$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

These unit vectors give an alternative way of expressing position vectors. If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then we have

$$\mathbf{v} = v_1\langle 1, 0, 0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

Extra Example

EXAMPLE 6 **Magnitudes and unit vectors** Consider the points $P(5, 3, 1)$ and $Q(-7, 8, 1)$.

a. Express $\overrightarrow{PQ}$ in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

b. Find the magnitude of $\overrightarrow{PQ}$.

c. Find the position vector of magnitude 10 in the direction of $\overrightarrow{PQ}$.

SOLUTION

a. $\overrightarrow{PQ}$ is equal to the position vector $\langle -7 - 5, 8 - 3, 1 - 1 \rangle = \langle -12, 5, 0 \rangle$. Therefore, $\overrightarrow{PQ} = -12\mathbf{i} + 5\mathbf{j}$.

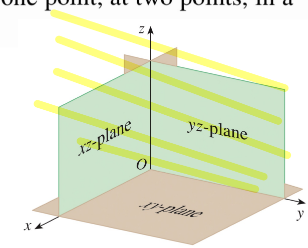
b. $|\overrightarrow{PQ}| = |-12\mathbf{i} + 5\mathbf{j}| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$

c. The unit vector in the direction of $\overrightarrow{PQ}$ is $\mathbf{u} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \dfrac{1}{13}\langle -12, 5, 0 \rangle$. Therefore, the vector in the direction of $\mathbf{u}$ with a magnitude of 10 is $10\mathbf{u} = \dfrac{10}{13}\langle -12, 5, 0 \rangle$.

Group Quiz Solutions

Consider the sphere $(x-3)^2 + (y-5)^2 + (z-4)^2 = 25$ S

(a) Does the sphere intersect each of the following planes at zero points, at one point, at two points, in a line, or in a circle?



i. The sphere intersects the yz-plane [?]

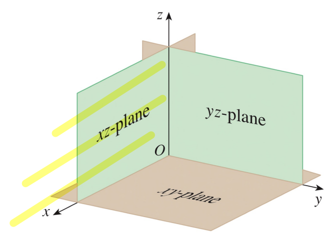
- yz-plane is the collection of points $(0, y, z)$
- Sub $x=0$ into S: $(-3)^2 + (y-5)^2 + (z-4)^2 = 25$

$$\left.(y-5)^2 + (z-4)^2 = 16\right\} \text{ a circle}$$

The sphere intersects the yz-plane

✓ $\boxed{\text{in a circle}}$

ii. The sphere intersects the xz-plane [?]



Answer: The xz-plane is the set of points $(x, 0, z)$ for any x, z.

Set : $(x-3)^2 + (-5)^2 + (z-4)^2 = 25$

$$(x-3)^2 + (z-4)^2 = 0$$

This equation is satisfied for $x = 3$, $z = 4$

So the sphere intersects the xz-plane at exactly $\boxed{\text{one point.}}$

This point is $(x=3, y=0, z=4)$

Group Quiz Solutions

Consider the sphere $(x-3)^2+(y-5)^2+(z-4)^2=25$

(b) Does the sphere intersect each of the following coordinate axes at zero points, at one point, at two points, or in a line?

The sphere intersects the z-axis $\boxed{?}$

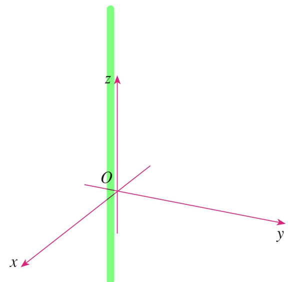


FIGURE 1
Coordinate axes

The z-axis  is the set of points $(0,0,z)$, z any number

Set $x=y=0$: $(-3)^2+(-5)^2+(z-4)^2=25$

$$(z-4)^2=-9$$

No z satisfies this equation

The sphere does <u>not</u> intersect the z-axis