News

- This coming Mon : Quiz 2 }
The following Wed: Quiz 3 } See Blackboard for which MML to study

- Before next class, read:

  Textbook Sec 13.5 "Lines & planes in space"

  - Example 1 or 2 (lines)
  - Example 5 or 6 (planes)
  - Example 8 or 9 (parallel or orthogonal planes)

12.1 Parametric equations

Idea: A turtle moves along a curve in 2D.
(called parametric curve)

The position (x, y) of the turtle can be described by

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\}$$ called (a set of) parametric equations

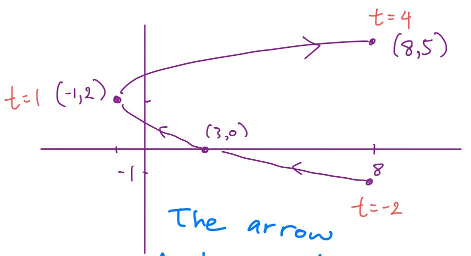
where both $f(t), g(t)$ are functions of time t (called a parameter)

Ex 1: Sketch and identify the curve defined by the parametric equations $\begin{cases} x = t^2 - 2t \\ y = t + 1 \end{cases}$ for $-2 \le t \le 4$

(a) Sketch, (b) identify

a) Plot

t	(x, y)
-2	(8, -1)
-1	(3, 0)
0	(0, 1)
1	(-1, 2)
2	
3	
4	(8, 5)



The arrow tells you the "positive orientation"

Def
The positive orientation is the direction in which a parametric curve is created as the parameter increases

b) Sometimes it's possible to eliminate the parameter (t) from parametric equations:

$y = t + 1 \Rightarrow t = y - 1$

(Plug into $x = t^2 - 2t$) $\quad x = (y-1)^2 - 2(y-1) \Rightarrow \boxed{x = y^2 - 4y + 3}$

So the curve is a parabola (coming from quadratic function) $\square$

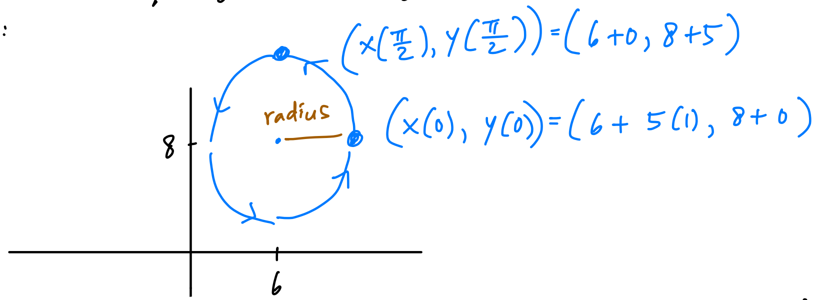
Circles

Ex 2: (Like MML #6)

a) Describe the positive orientation of the curve given by the parametric equations

$$\left.\begin{aligned} x &= 6 + 5\cos t \\ y &= 8 + 5\sin t \end{aligned}\right\} \text{ for } 0 \le t \le 2\pi$$

Sol:



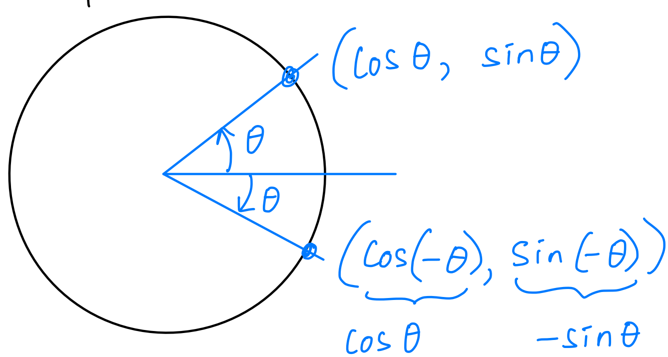
Description: As t goes from 0 to 2π, we trace the circle centered at $(6,8)$ w/ radius 5 once, in the counterclockwise direction, starting from $(11,8)$

b) The parametric equations

$$\left.\begin{aligned} x &= 6 + 5\cos(-t) \\ y &= 8 + 5\sin(-t) \end{aligned}\right\} \quad 0 \le t \le 6\pi$$

trace the same circle three times in the clockwise direction, starting at point $(11,8)$ also

because ...



c) In general, the curve

$$\left.\begin{aligned} x &= x_0 + R\cos\left(\frac{2\pi}{b}t\right) \\ y &= y_0 + R\sin\left(\frac{2\pi}{b}t\right) \end{aligned}\right\} \text{ for } 0 \le t \le b$$

here b is a positive number

traces the circle with radius R centered at (x_0, y_0) ONCE, in counterclockwise direction.

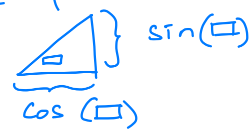
d) In general, in this situation we can always remove the parameter:

$$(x-x_0)^2 = R^2\cos^2(\square)$$
$$(y-y_0)^2 = R^2\sin^2(\square)$$

$+$

$$(x-x_0)^2 + (y-y_0)^2 = R^2\left[\underbrace{\cos^2(\square) + \sin^2(\square)}\right]$$

equal to 1 because of the Pythagorean Thm:



The familiar equation in x & y for the circle with radius R centered at (x_0, y_0) is

$$\boxed{(x-x_0)^2 + (y-y_0)^2 = R^2}$$

See Sec 13.2.

Lines

* The parametric equations

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \end{aligned}\right\} \text{ for } -\infty < t < \infty$$

describe the line passing through $\underbrace{(x_0, y_0)}_{\text{when } t=0}$, w/ slope $\frac{b}{a}$ (if $a \neq 0$)

- Here the $\underbrace{\text{positive orientation}}_{\text{(direction as } t \text{ increases)}}$ is

$$\begin{cases} \text{left to right} & \text{if } a > 0 \\ \text{the opposite direction} & \text{if } a < 0 \end{cases}$$

- In this situation, we can eliminate the parameter to get ...

$$x = x_0 + at \Rightarrow \frac{x - x_0}{a} = t$$

Sub this first equation into $y - y_0 = bt$:

$$y - y_0 = b\,\frac{x - x_0}{a}$$

$$\boxed{y - y_0 = \frac{b}{a}(x - x_0)}$$ an equation in x & y

* The parametric equations $\left.\begin{aligned} x &= x_0 \\ y &= y_0 + bt \end{aligned}\right\}$ for $-\infty < t < \infty$

describes a vertical line ("slope" is undefined) passing through the point (x_0, y_0).

- Positive orientation is toward north if $b > 0$
- Equation in x & y w/o the parameter is $\boxed{x = x_0}$

Ex 3: (like MML #3)

Consider parametric equations

$x = -2 + 3t$
$y = 4 - 6t$ for $-\infty < t < \infty$

a) Eliminate the parameter to obtain an equation in x and y

Sol: Solve first eq for t:

$x + 2 = 3t$

$\frac{x+2}{3} = t$

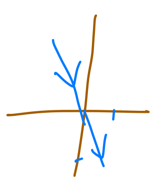
Replace t in the second eq w/ $\frac{x+2}{3}$

$y = 4 - 6\left(\frac{x+2}{3}\right)$

$y = 4 - 2x + 4 \Rightarrow \boxed{y = -2x}$

b) sketch parametric curve & indicate pos orientation:

Sol:



(because "3" is positive, positive orientation is left to right)

Ex 4: (like MML # 8 & 9)

Find a set of parametric equations for the line segment from point $P(4,7)$ to $Q(2,-3)$.

Sol: • Slope of this line is $\frac{y_2-y_1}{x_2-x_1}$ (or $\frac{y_1-y_2}{x_1-x_2}$ same number)

which is $\frac{7-(-3)}{4-2} = \frac{10}{2} = 5$.

• Now choose a, b so that $\frac{b}{a} = 5$, e.g. choose $a=1$, $b=5$.

• Then the parametric equations

$$(*) \qquad \left.\begin{aligned} x &= 4+t \\ y &= 7+5t \end{aligned}\right\} \text{ for } t \in (-\infty, \infty)$$

give the line w/ slope 5 passing through $P(4,7)$ from left to right.

• However, what we want is the line segment from $\underset{x=4}{P(4,7)}$ to $\underset{x=2}{Q(2,-3)}$, right to left.

• So replace our a, b in $(*)$ w/ $a=-1$, $b=-5$: $\boxed{\begin{aligned} x &= 4-t \\ y &= 7-5t \end{aligned}}$

• Next, find the correct interval:

The starting point of t should be 0 so that we have $P(4,7)$ at the beginning.

The ending point of t should be when

$$\left.\begin{aligned} x(t) &= 2 \\ y(t) &= -3 \end{aligned}\right\} \qquad \left.\begin{aligned} 2 &= 4-t \\ -3 &= 7-5t \end{aligned}\right\} \Rightarrow t=2$$

So the interval is $\boxed{0 \le t \le 2}$

Note: We can choose other b, a such that $\frac{b}{a} = 5$, so there are infinitely many possible answers. □