Additional Review Problems 2, 6—10

2. Find a normal vector to the surface $xy - z^2 = y^2 z - 1$ at the point $\underbrace{(7, 3, 2)}_{= P_0}$. Then find the equation of the tangent plane at this point.

Move all variables to one side: $0 = -xy + z^2 + y^2 z - 1$

Set $F(x,y,z) := -xy + z^2 + y^2 z - 1$

$F_x = -y$, $F_y = -x + 2yz$, $F_z = 2z + y^2$

$F_x(P_0) = -3$, $F_y(P_0) = -7 + 2(3)(2) = 5$, $F_z(P_0) = 4 + 3^2 = 13$

$\nabla F(P_0) = \boxed{\langle -3, 5, 13 \rangle}$ or any nonzero scalar multiple is a normal vector to $0 = F(x,y,z)$

Equation of the tangent plane at $P_0(7,3,2)$ is

$$\boxed{-3(x-7) + 5(y-3) + 13(z-2) = 0}$$ or

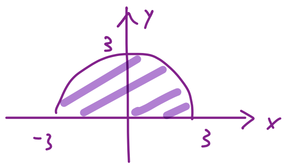
$$\boxed{-3x + 5y + 13z = 20}$$

6. Convert the following double integral from Cartesian to polar coordinates. Then evaluate the integral:

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(\underbrace{\pi x^2+\pi y^2}_{\pi(x^2+y^2)=\pi r^2})\,dy\,dx$$

$R = \{(x,y) : -3 \le x \le 3,\ 0 \le y \le \sqrt{9-x^2}\}$ in Cartesian

Sketch of R:



$R = \{(r,\theta) : 0 \le r \le 3,\ 0 \le \theta \le \pi\}$ in polar

Iterated integral in polar:

$$\int_0^{\pi}\int_0^{3} \sin(\pi r^2)\, \underbrace{r}_{\text{extra}}\, dr\, d\theta$$

inner $\displaystyle\int_0^3 r\sin(\pi r^2)\,dr = \frac{1}{2}\int_{u=0}^{u=9}\sin(\pi u)\,du = -\frac{1}{2}\frac{1}{\pi}\cos(\pi u)\Big|_{u=0}^{u=9}$

$u = r^2$

$du = 2r\,dr$

$\frac{1}{2}du = r\,dr$

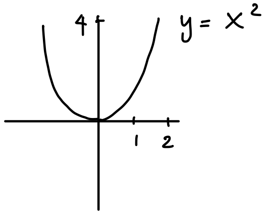
$\displaystyle = -\frac{1}{2\pi}\left[\cos(\pi 9) - \cos(0)\right] = -\frac{1}{2\pi}\left[-1-1\right] = \frac{1}{\pi}$

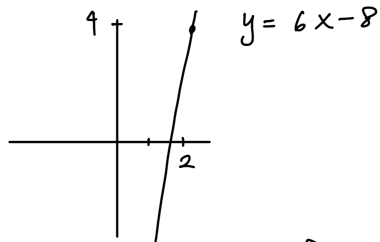
outer $\displaystyle\int_0^{\pi}\frac{1}{\pi}\,d\theta = \frac{1}{\pi}\theta\Big|_{\theta=0}^{\theta=\pi} = \frac{\pi}{\pi} - \frac{0}{\pi} = \boxed{1}$

the end of Q6

7. Consider the region R in the xy-plane bound by the curves $y = x^2$ and $y = 6x - 8$. Let D be the solid lying above the region R and below the plane $z = x$.

Set up:





The two curves intersect in the first quadrant:

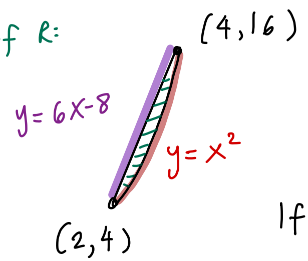
Set $y = x^2$ and $y = 6x-8$ equal:

$$x^2 = 6x - 8$$

$$x^2 - 6x + 8 = 0$$

$$(x-2)(x-4) = 0 \qquad x = 2, 4$$

Sketch of R:



I can think of the two curves as left/right bounds or as lower/upper bounds.

If I think of them as lower/upper bounds, I get $R = \{(x,y) : 2 \le x \le 4 \;,\; x^2 \le y \le 6x-8\}$.

So $\displaystyle\iint_R f(x,y)\, dA = \int_2^4 \int_{x^2}^{6x-8} f(x,y)\, dy\, dx$

cont →

7. Consider the region R in the xy-plane bound by the curves $y = x^2$ and $y = 6x - 8$. Let D be the solid lying above the region R and below the plane $z = x$.

(a) Use the appropriate double integral to compute the volume of the solid D.

Because R lives in the xy-plane $(z=0)$, this phrase tells us D is bounded below by $z=0$.

This tells us D is bounded by $z = x$

$$D = \{ (x,y,z) : 2 \le x \le 4,\ x^2 \le y \le 6x-8,\ \underset{\text{lower}}{0} \le z \le \underset{\text{upper}}{x} \}$$

a) Volume of D is $\displaystyle \iint_R \underset{\text{upper}}{x} - \underset{\text{lower}}{0} \; dA = \int_2^4 \int_{x^2}^{6x-8} x \; dy\, dx$

inner $\displaystyle \int_{x^2}^{6x-8} x \; dy = xy \Big|_{y=x^2}^{y=6x-8} = x(6x-8-x^2) = 6x^2 - 8x - x^3$

outer $\displaystyle \int_2^4 6x^2 - 8x - x^3 \; dx = \frac{6x^3}{3} - \frac{8x^2}{2} - \frac{x^4}{4} \Big|_{x=2}^{x=4}$

$$= 2(4^3) - 4(4^2) - \frac{4^4}{4} - \left[2(2^3) - 4(2^2) - \frac{2^4}{4} \right] = 4$$

Volume of D is $\boxed{4}$

(b) Use the appropriate triple integral to compute the volume of the solid D.

b) Volume of D is $\displaystyle \iiint_D 1 \, dV = \int_2^4 \int_{x^2}^{6x-8} \int_0^x 1 \; dz\, dy\, dx$

inner $\displaystyle \int_0^x 1 \; dz = z \Big|_{z=0}^{z=x} = x$ middle $\displaystyle \int_{x^2}^{6x-8} x \; dy = 6x^2 - 8x - x^3$

outer $\displaystyle \int_2^4 6x^2 - 8x - x^3 \; dx = \boxed{4}$ same as in part (a)

con't →

(c) Find the average value of the function $f(x, y) = x$ over the region R.

Average value of $f(x,y)$ over the region R is

$$\frac{1}{\text{area of } R} \iint_R f(x,y)\, dA.$$

Area of R is $\iint_R 1\, dA = \int_2^4 \int_{x^2}^{6x-8} 1\, dy\, dx$

<u>inner</u> $\int_{x^2}^{6x-8} 1\, dy = y \Big|_{y=x^2}^{y=6x-8} = 6x-8-x^2$

<u>outer</u> $\int_2^4 6x-8-x^2 \; dx = 6\frac{x^2}{2} - 8x - \frac{x^3}{3} \Big|_{x=2}^{x=4}$

$$= 3(4^2)-8(4)-\frac{4^3}{3} - \left[3(2^2)-8(2)-\frac{2^3}{3}\right] = \frac{4}{3}$$

$\boxed{\text{Area of } R \text{ is } \frac{4}{3}}$

$\iint_R f(x,y)\, dA$ for $f(x,y) = x$ is $\iint_R x\, dA = 4$. We computed this in part (a).

So $\frac{1}{\text{area of } R} \iint_R f(x,y)\, dA = \frac{1}{\left(\frac{4}{3}\right)} 4 = \boxed{3}$ is the average value of $f(x,y) = x$ over R.

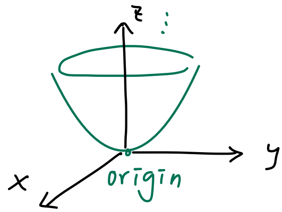
<u>end of Q7</u>

Call this D_1

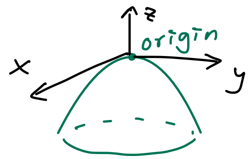
8. Consider the solid below the paraboloid $z = 16 - x^2 - y^2$ and above the xy-plane. A cylindrical hole is cut through this solid using the cylinder $x^2 + y^2 = 4$, resulting in a new solid D. Set up a double integral in polar coordinates for computing the volume of the solid D, then compute the volume.

Sketch D_1 :

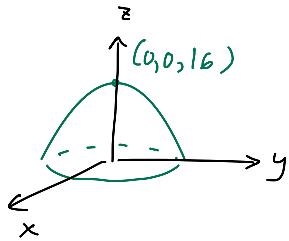
the paraboloid $z = x^2 + y^2$ is



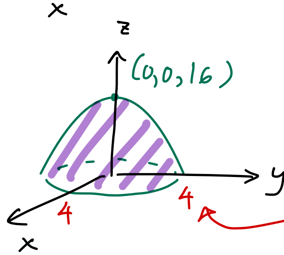
$z = -(x^2+y^2)$ is



the paraboloid $z = 16 - (x^2 + y^2)$ is



Sketch of the solid D_1 is



D_1 = all points below $z = 16 - x^2 - y^2$ and above the xy-plane $(z=0)$

The intersection of $z = 16 - x^2 - y^2$ and $z = 0$ is a circle C centered at the origin. To find this circle C, set $z = 16 - x^2 - y^2$ and $z=0$ equal: $0 = 16 - x^2 - y^2$

$$x^2 + y^2 = 4^2 \text{ or } r = 4$$ (circle with radius 4)

So D_1 is bounded below by the disk $R_1 = \{(r,\theta) : 0 \le r \le 4,\ 0 \le \theta \le 2\pi\}$ (in the xy-plane)

and D_1 is bounded above by the surface

$$z = 16 - (x^2+y^2) \text{ or } z = 16 - r^2$$

Con't →

$$D_1 = \{(r,\theta,z) : \underbrace{0 \le r \le 4, 0 \le \theta \le 2\pi}_{\text{disk } R_1}, \underset{\text{xy-plane}}{0} \le z \le \underset{\text{paraboloid}}{\underline{16 - r^2}}\}$$

Call this D_1

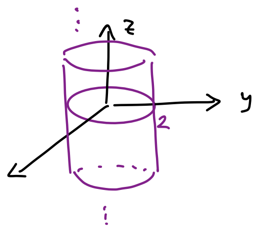
8. Consider the solid below the paraboloid $z = 16 - x^2 - y^2$ and above the xy-plane. A cylindrical hole is cut through this solid using the cylinder $x^2 + y^2 = 4$, resulting in a new solid D. Set up a double integral in polar coordinates for computing the volume of the solid D, then compute the volume.

Call this D_2

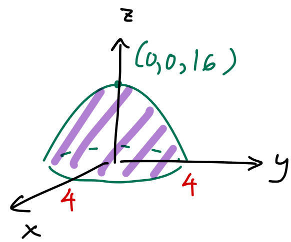
D_2 is the surface $x^2 + y^2 = 4$

or

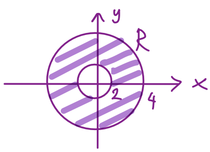
$r = 2$



Cutting cylinder D_2 through solid D_1



we get $D = \{(r,\theta,z) : \underbrace{2 \le r \le 4, 0 \le \theta \le 2\pi}_{\text{washer } R}, \underset{\text{xy-plane}}{0} \le z \le \underset{\text{paraboloid}}{\underline{16 - r^2}}\}$



$R = \{(r,\theta) : 2 \le r \le 4, 0 \le \theta \le 2\pi\}$

$$\boxed{\text{Volume of } D = \iint_R \underset{\text{paraboloid}}{(16 - r^2)} - \underset{\text{xy-plane}}{0} \, dA = \int_0^{2\pi} \int_2^4 (16 - r^2)\, \overset{\text{extra}}{r} \, dr \, d\theta}$$

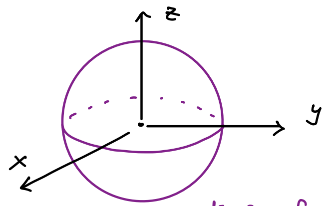
<u>inner</u> $\displaystyle\int_2^4 16r - r^3 \, dr = 16\frac{r^2}{2} - \frac{r^4}{4} \Big|_{r=2}^{r=4} = 8(4^2) - \frac{4^4}{4} - \left(8(2^2) - \frac{2^4}{4}\right) = 36$

<u>outer</u> $\displaystyle\int_0^{2\pi} 36 \, d\theta = 36(2\pi) = \boxed{72\pi}$ is the volume of D.

<u>end of Q8</u>

9. Consider the solid D bound by the sphere $x^2 + y^2 + z^2 = 20$ and the paraboloid $z = x^2 + y^2$ in the first octant. Set up a triple integral in cylindrical coordinates to compute the volume of this solid.

Sketch of the sphere $x^2 + y^2 + z^2 = 20$:



radius of sphere is $\sqrt{20}$

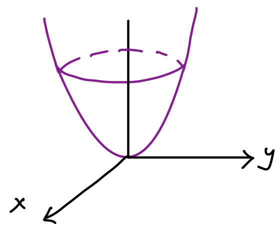
$\frac{1}{4}$ of the upper hemisphere:

$z = \sqrt{20 - (x^2 + y^2)}$ where $x \geq 0$, $y \geq 0$

In cylindrical coordinates:

$z = \sqrt{20 - r^2}$

Sketch of the paraboloid $z = x^2 + y^2$:



In cylindrical coordinates: $z = r^2$

Find the curve C where these two surfaces intersect:

Set $r^2 = \sqrt{20 - r^2}$

$r^4 = 20 - r^2$

$r^4 + r^2 - 20 = 0$

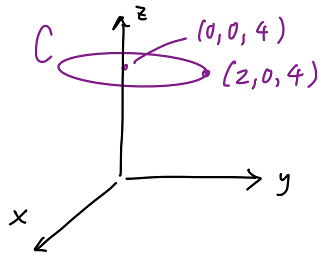
$(r^2 + 5)(r^2 - 4) = 0$

$r^2 = -5$ (not possible) or $r^2 = 4$

$\boxed{r = 2 \quad \text{and} \quad z = 4}$

This is the equation for C

So the intersection C is the circle centered at $(0,0,4)$ with radius 2, living in the plane $z = 4$:



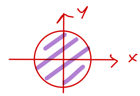
Con't →

9. Consider the solid D bound by the sphere $x^2+y^2+z^2=20$ and the paraboloid $z=x^2+y^2$ in the first octant. Set up a triple integral in cylindrical coordinates to compute the volume of this solid.

So the solid bounded by the upper hemisphere $z=\sqrt{20-r^2}$ and the paraboloid $z=r^2$ is

$$\left\{(r,\theta,z):\ \underbrace{0\le r\le 2,\ 0\le\theta\le 2\pi}_{\text{disk centered at the origin with radius 2, in the xy-plane}},\ r^2\le z\le\sqrt{20-r^2}\right\}$$

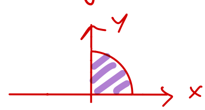
disk centered at the origin with radius 2, in the xy-plane



Since our solid D is in the first (positive octant), we require x and y to be nonnegative, so

$$D=\left\{(r,\theta,z):\ \underbrace{0\le r\le 2,\ 0\le\theta\le \tfrac{\pi}{2}}_{\text{disk centered at the origin with radius 2, in the xy-plane, first quadrant}},\ r^2\le z\le\sqrt{20-r^2}\right\}$$

disk centered at the origin with radius 2, in the xy-plane, first quadrant



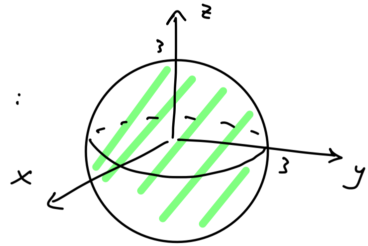
Volume of D is $\iiint_D 1\,dV = \boxed{\int_0^{\frac{\pi}{2}}\int_0^2\int_{r^2}^{\sqrt{20-r^2}} \underset{(\text{extra})}{r}\,dz\,dr\,d\theta}$

end of Q9

10. Let D be the top half of a ball of radius 3 centered at the origin. Find the average distance of points in D from the origin using the appropriate triple integral in spherical coordinates.

Hint: Set up a function $f(\rho, \varphi, \theta)$ which gives the distance of the point (ρ, φ, θ) to the origin. Then use the triple integral formula for the average of a function.

Sketch D:



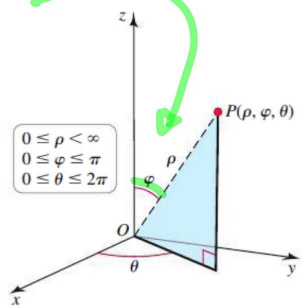
ball of radius 3 centered at the origin:

$\{(x,y,z): x^2+y^2+z^2 \le 3^2\}$ or

$\{(\rho,\varphi,\theta): 0 \le \rho \le 3,\ 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi\}$

rho phi theta

φ is the angle between the positive z-axis and the line from the origin to a point:



Since D is the top half of the ball, we need to restrict φ to be between 0 and $\frac{\pi}{2}$.

So $D = \{(\rho,\varphi,\theta): 0 \le \rho \le 3,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi\}$.

$\rho^2 \sin\varphi$ extra

Volume of D is $\iiint_D 1\, dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\underbrace{\int_0^3 \rho^2\, d\rho}_{\text{inner}} \sin\varphi\, d\varphi\, d\theta$

inner $\int_0^3 \rho^2\, d\rho = \frac{\rho^3}{3}\Big|_{\rho=0}^{\rho=3} = \frac{3^3}{3} - 0 = 9$

middle $\int_0^{\frac{\pi}{2}} 9 \sin\varphi\, d\varphi = -9\cos\varphi \Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}} = -9\left(\cos\frac{\pi}{2} - \cos 0\right) = -9(-1) = 9$

outer $\int_0^{2\pi} 9\, d\theta = 9(2\pi) = \boxed{18\pi}$ is the volume of D. con't $\longrightarrow$

10. Let D be the top half of a ball of radius 3 centered at the origin. Find the average distance of points in D from the origin using the appropriate triple integral in spherical coordinates.

Hint: Set up a function $f(\rho, \varphi, \theta)$ which gives the distance of the point (ρ, φ, θ) to the origin. Then use the triple integral formula for the average of a function.

the distance of a point $P(\rho, \varphi, \theta)$ from the origin is ρ (by definition). So let $f(\rho,\varphi,\theta) := \rho$.

Average distance of points in D from the origin is average value of our function $f(\rho,\varphi,\theta) = \rho$ over D

which is $\dfrac{1}{\text{volume of } D} \iiint_D f(\rho,\varphi,\theta)\, dV.$

In the previous page, volume of D was found to be 18π.

$\rho^2 \sin\varphi$ extra

Now, compute $\iiint_D \rho \, dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \underbrace{\int_0^3 \rho\, \rho^2 \, d\rho}_{\text{inner}} \sin\varphi \, d\varphi \, d\theta$

inner $\int_0^3 \rho^3 \, d\rho = \dfrac{\rho^4}{4}\Big|_{\rho=0}^{\rho=3} = \dfrac{3^4}{4} - 0 = \dfrac{81}{4}$

middle $\int_0^{\frac{\pi}{2}} \dfrac{81}{4} \sin\varphi \, d\varphi = -\dfrac{81}{4}\cos\varphi \Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}} = -\dfrac{81}{4}\left(\cos\frac{\pi}{2} - \cos 0\right) = -\dfrac{81}{4}(-1) = \dfrac{81}{4}$

outer $\int_0^{2\pi} \dfrac{81}{4} d\theta = \dfrac{81}{4}(2\pi) = \boxed{\dfrac{81}{2}\pi} = \iiint_D \rho \, dV$

$\dfrac{1}{\text{volume of } D} \iiint_D f(\rho,\varphi,\theta)\, dV = \dfrac{1}{(18\pi)} \dfrac{81}{2}\pi = \boxed{\dfrac{9}{4}}$ is

the average distance of points in D from the origin.

end of Q 10