9.1 Modeling with Differential Equations

Differential equations. An equation containing an unknown function and some of its derivatives is a *differential equation*. Examples are

$$\frac{dy}{dx} = 4x \quad \text{and} \quad y''(x) + x^3 y'(x) = xy(x).$$

The *order* of a differential equation is the order of the highest-order derivative in the equation. For example, $\frac{dy}{dx} = 4x$ is a first-order differential equation, while $y''(x) + x^3 y'(x) = xy(x)$ is a second-order differential equation.

Motivation. A mathematical model of a real-world problem (formulated through reasoning or based on data) often takes the form of a differential equation.

Models for population growth.

Variables:

i. Assuming ideal conditions, population grows at a rate proportional to the population size.

ii. Assume limited resources.

Logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Example:

(i) Check whether every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution to the differential equation $y' = \frac{1}{2}(y^2 - 1)$.

(ii) Find a solution to the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$.

Example: Which of the functions below satisfy the differential equation $y'' + y = \sin x$?

(a) $y = \sin x$

(b) $y = \cos x$

(c) $y = \frac{1}{2} x \sin x$

(d) $y = -\frac{1}{2} x \cos x$

Thinking about the problem:

First we will find y' and then y'' for each of the functions and then compute $y'' + y$ in each case to see if we get $\sin x$.

Doing the problem:

(a) $y = \sin x \Longrightarrow y'' = -\sin x$

$\Longrightarrow y'' + y = 0,$

(b) $y = \cos x \Longrightarrow y'' = -\cos x$

$\Longrightarrow y'' + y = 0,$

(c) $y = \frac{1}{2} x \sin x \Longrightarrow y'' = \cos x - \frac{1}{2} x \sin x$

$\Longrightarrow y'' + y = \cos x,$

(d) $y = -\frac{1}{2} x \cos x \Longrightarrow y'' = \sin x + \frac{1}{2} x \cos x$

$\Longrightarrow y'' + y = \sin x.$

The only solution to $y'' + y = \sin x$ among the four functions here is (d) $y = -\frac{1}{2} x \cos x$.

Solutions should show all of your work, not just a single final answer.

1. We consider the differential equation $\frac{dy}{dt} = 1 - 2y$. (Here the independent variable is t.)

(a) Find all constant solutions. That is, if $y = K$ for constant K satisfies the differential equation, what does K need to be?

(b) Show every function of the form $y(t) = \frac{1}{2} + Ce^{-2t}$, where C is a constant, is a solution of the differential equation.

(c) If $y(t)$ is a function described by part (b), what can you say about the long-term behavior $\lim_{t \to \infty} y(t)$?

2. We consider the differential equation $\dfrac{dy}{dx} = xy$. (Here the independent variable is x.)

   (a) Find all constant solutions.

   (b) Show every function of the form $y(x) = Ce^{x^2/2}$, where C is a constant, is a solution.

   (c) For a solution as in part (b), describe C as a value of $y(x)$.

3. True/False (give justification/ counterexample)

   Every differential equation has a constant solution.