GROWTH RATES

Definition: Growth Rates of Functions (as x approaches infinity)

Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then

- f *grows faster than* g as $x \to \infty$ if $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$.
- f and g *have comparable growth rates* if $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = M$, where M is a positive number.

Theorem: Asymptotic Hierarchy

Let $f \ll g$ mean that g grows faster than f as $x \to \infty$. Then

$$c \ll (\ln x)^q \ll x^p \ll a^x \ll x! \ll x^x$$

$$c \ll (\ln n)^q \ll n^p \ll a^n \ll n! \ll n^n$$

Example:
Show that $\dfrac{x^p}{\ln(x)} \to \infty$ as $x \to \infty$:

$$\begin{aligned} \lim_{x\to\infty} \frac{x^p}{\ln x} &= \lim_{x\to\infty} \\ &= \lim_{x\to\infty} \\ &= \end{aligned}$$

Example:
Show that r^x (for $r > 1$) grows faster than x^p:

$$\begin{aligned} \lim_{x\to\infty} \qquad\qquad &= \lim_{x\to\infty} \\ &= \lim_{x\to\infty} \\ &= \end{aligned}$$

Review:
$\dfrac{d}{dx}(4^x) = \boxed{?}$

$$c \ll (\ln n)^q \ll n^p \ll a^n \ll n! \ll n^n$$

$$\lim_{n\to\infty} \sqrt[n]{c} \qquad \lim_{n\to\infty} \sqrt[n]{(\ln n)^q} \qquad \lim_{n\to\infty} \sqrt[n]{n^p} \qquad \lim_{n\to\infty} \sqrt[n]{a^n} \qquad \lim_{n\to\infty} \sqrt[n]{n!} \qquad \lim_{n\to\infty} \sqrt[n]{n^n}$$

$$\boxed{\lim_{n\to\infty} c^{\frac{1}{n}}} =$$

$$=$$

$$\boxed{\lim_{n\to\infty} (\ln n)^{\frac{q}{n}}} = \boxed{?}$$

Let $y =$

$\ln y =$

$\boxed{\lim_{n\to\infty} (n)^{\frac{p}{n}}} = \boxed{?}$

Let $y =$

$\ln y =$

$\boxed{\lim_{n\to\infty} (n!)^{\frac{1}{n}}} = \boxed{?}$

$$\begin{aligned}
\text{Let } y &= (n!)^{\frac{1}{n}} \\
\ln(y) &= \frac{1}{n}\ln(n!) \\
&= \frac{1}{n}\ln\big((1)\,.(2)\,.(3)\ldots(n-2)\,.(n-1)\,.(n)\big) \quad \text{by definition of factorial} \\
&= \frac{1}{n}\big(\ln(1) + \ln(2) + \ln(3) + \cdots + \ln(n-2) + \ln(n-1) + \ln(n)\big) \quad \text{by log laws} \\
&\geq \frac{1}{n}\int_1^n \ln x\,dx \quad \text{Why? Will cover in Sec 7.8: improper integrals and Sec 11.3: Estimates of Sum.}
\end{aligned}$$

To recap, the above sequence of equality and inequality symbols shows us that

$$\underline{\hspace{3cm}} \geq \underline{\hspace{4.5cm}}.$$

$$\begin{aligned}
\text{Since } \lim_{n\to\infty}\frac{1}{n}\left(\int_1^n \ln x\,dx\right) &= \lim_{n\to\infty}\frac{1}{n}\left(x\,\ln(x)\Big|_{x=1}^{x=n} - \int_1^n dx\right) \quad \text{using integration by parts (will be covered in Calc II)} \\
&= \lim_{n\to\infty}\frac{1}{n}(n\,\ln(n) - n + 1) \\
&= \lim_{n\to\infty} \\
&= \underline{\hspace{3cm}},
\end{aligned}$$

we can conclude that $\lim_{n\to\infty}\ln(y) = \infty$ as well. Therefore

$$\begin{aligned}
\lim_{n\to\infty}(n!)^{\frac{1}{n}} &= \lim_{n\to\infty} e^{\ln(y)} \\
&= \underline{\hspace{3cm}}.
\end{aligned}$$

USING GROWTH RATES TO DETERMINE WHETHER A SERIES CONVERGES

Refer to the "growth rates" notes.

1. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\ln\ (n+1)}$$

(a) The denominator of the term is *ln* $(n+1)$. Consider the function $\ln(x+1)$. Fill in the blank with $\ll$ or $\gg$:

$$\ln(x+1) \quad \underline{\qquad} \quad x.$$

(b) This means that

$$\lim_{x\to\infty} \underline{\qquad\qquad} = \infty$$

(See page 1 of this "growth rates" notes)

(c) Fill in the blank with $\leq$ or $\geq$:

$$\ln(n+1) \quad \underline{\qquad} \quad n \qquad \text{for large enough } n.$$

(d) Fill in the blank with $\leq$ or $\geq$:

$$\frac{1}{\ln(n+1)} \quad \underline{\qquad} \quad \frac{1}{n} \qquad \text{for sufficiently large } n.$$

(e) State the test which you would use to determine whether the series $\sum \frac{1}{n}$ converges or diverges.

(f) State whether the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

(g) State the statement of the limit comparison test.

(h) Compute $\lim_{n\to\infty} \frac{n}{\ln(n+1)}$. Note that the answer is given in part (b).

(i) By the limit comparison test and part (f), the series

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)} \quad \underline{\qquad\qquad\qquad}$$

2. Let p be any positive integer (say, $p = 5$) Let a be a number larger than 1 (say, $a = \frac{3}{2}$). Consider the series

$$\sum_{n=1}^{\infty} \frac{n^p}{a^n} \quad = \quad \sum_{n=1}^{\infty} \frac{n^5 2^n}{3^n}.$$

After simplifying, we realize that the term of the series is

$$\frac{n^5}{(\frac{3}{2})^n}.$$

(a) The series $\sum_{n=1}^{\infty} n^5$ ____________ by the ________________________ test.

(b) The series $\sum_{n=1}^{\infty} (\frac{2}{3})^n$ ____________ by the ____________________ test.

(c) Let's figure out: is the numerator/ denominator more dominant than the other?

(d) The denominator of the term is $\left(\frac{3}{2}\right)^n$. Consider the function $\left(\frac{3}{2}\right)^x$.
The numerator of the term is n^5. Consider the function x^5.
Fill in the blank with $\ll$ or $\gg$:

$$x^5 \quad \_\_\_\_ \quad \left(\tfrac{3}{2}\right)^x.$$

(e) This means that (check the "growth rates" notes)

$$\lim_{x\to\infty} \_\_\_\_\_\_\_\_ = \infty. \text{ This also means } \lim_{x\to\infty} \_\_\_\_\_\_\_\_ = 0.$$

(f) Part (e) means that __________________ is more dominant than ________________.

(g) Quiz your classmate on the statement of the ratio test until it's memorized.

(h) (Recall that when you see *only* polynomial-like terms, like $n^{p_1} + n^{p_2}$, the ratio test *will be inconclusive* (*Why?*). But, if you see powers like a^n it's OK to use the ratio test. Let's apply the ratio test to $\sum a_n = \sum \frac{n^5 2^n}{3^n}$. Compute

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^5}{\left(\frac{3}{2}\right)^{(n+1)}} \frac{\left(\frac{3}{2}\right)^n}{n^5}$$

$$=$$

(i) By the ratio test, the series

$$\sum_{n=1}^{\infty} \frac{n^5 2^n}{3^n} \quad \_\_\_\_\_\_\_\_\_\_\_\_.$$

3. Consider the series

$$\sum_{n=1}^{\infty} a_n$$

for

$$a_n = \frac{n^n}{7^n\ (n)!} \qquad \text{and} \qquad a_n = \frac{n^n}{2^n\ (n)!}$$

(a) Look at the term of the series.

The numerator, n^n looks like the function ________.

The denominator, $7^n\ (n)!$ looks like the product of functions ____________ and _____________.

(b) Which is more dominant for large n? The numerator or the denominator? Can you tell just by looking at the "growth rates" notes?

(c) I told you that the ratio test *will probably work* if you see exponents like r^n or a factorial. Compute

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

for

$$a_n = \frac{n^n}{7^n\ n!} \qquad \text{and} \qquad a_n = \frac{n^n}{2^n\ n!}$$

(Recall that $(1 + \frac{1}{n})^n \to e$ as $n \to \infty$)

(d) By the ratio test, the series

$$\sum \frac{n^n}{7^n\ n!}\ \underline{\hspace{4cm}} \qquad \text{and} \qquad \sum \frac{n^n}{2^n\ n!}\ \underline{\hspace{4cm}}.$$