

Example:

Suppose $f(x)$ is a continuous and positive function on $[1, \infty)$.

a. Use the **Right Endpoint Rule** with $n = 5$ to approximate the integral $\int_1^6 f(x) dx$.

b. Use the **Left Endpoint Rule** with $n = 5$ to approximate the integral $\int_1^6 f(x) dx$.

c. Suppose $f(x)$ is **decreasing**, then (fill in $<$, $=$ or \geq)

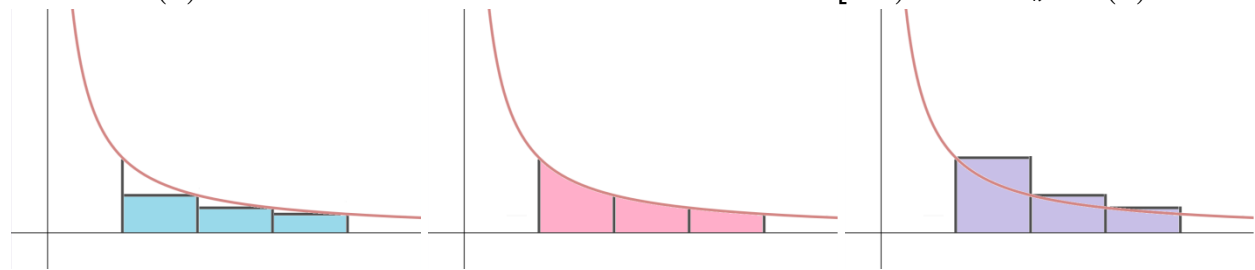
the estimated value in part (a) _____ the value of $\int_1^6 f(x) dx$ and

the estimated value in part (b) _____ the value of $\int_1^6 f(x) dx$.

Answer on the next page (don't go to the next page yet)

Integral Test

Suppose $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then



$$a_2 + a_3 + a_4 \leq \int_1^4 f(x) dx \leq a_1 + a_2 + a_3$$

In general,

$$\sum_{k=2}^n a_k \leq \int_1^n f(x) dx \leq \sum_{k=1}^{n-1} a_k$$

The Integral Test

Suppose f is a **continuous, positive, decreasing** function on $[1, \infty)$ and let $a_n = f(n)$. Then

- If $\int_1^{\infty} f(x) dx$ is **convergent**, then $\sum_{n=1}^{\infty} a_n$ is _____.
- If $\int_1^{\infty} f(x) dx$ is **divergent**, then $\sum_{n=1}^{\infty} a_n$ is _____.

When we use the Integral Test

- It is not necessary to start the series or the integral at $n = 1$. For example, in testing the series $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$ we can use $\int_4^{\infty} \frac{1}{(x-3)^2} dx$.

- It is not necessary that f be **always** decreasing. What is important is that f be **ultimately** decreasing. That is, decreasing on $[N, \infty)$ for some number N . Then $\sum_{n=N+1}^{\infty} a_n$

is convergent, which means $\sum_{n=1}^{\infty} a_n$ is convergent.

We should **NOT** infer from the Integral Test that the sum of the series is equal to the value of the integral. In general,

$$\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x) dx.$$

Useful Fact

1. A **continuous** function is continuous at every point on its **domain**.
 - Polynomials/Root functions/Trig functions/Exponential functions/Log functions are continuous functions.
 - If f and g are continuous at a , then $\frac{f}{g}$ is continuous at a provided $g \neq 0$.
2. If $f'(x) < 0$ on the interval (a, b) , then $f(x)$ is **decreasing** on the interval (a, b) .

Example: Use the **Integral Test** to determine the convergence or divergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

The Integral Test

Proof of the p -series test

Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

If $p < 0$, then $\lim_{n \rightarrow \infty} \frac{1}{n^p} = \infty$. If $p = 0$, then $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 1$.

In either case, $\lim_{n \rightarrow \infty} \frac{1}{n^p} \neq 0$, so the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by the Divergence Test.

If $p > 0$, then the function $f(x) = \frac{1}{x^p}$ is continuous, positive and decreasing on $[1, \infty)$.

We showed earlier (Sec 7.8 part 1 notes): $\int_1^{\infty} \frac{1}{x^p} dx$ converges if $p > 1$ and diverges if $p \leq 1$.

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$ by the Integral Test.

 p -series

The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is

convergent if _____ and divergent if _____.

Practice/Review:

Determine whether the series $\sum_{k=1}^{\infty} k^{-\frac{3}{4}}$ converges or diverges.

Practice/Review:

Determine whether the series $\sum_{k=4}^{\infty} \frac{1}{(k-1)^{\sqrt{2}}}$ converges or diverges.

Practice/Review:

Which of the following is a convergent p -series?

A.) $\sum_{k=1}^{\infty} \frac{3}{2^k}$

B.) $\sum_{k=1}^{\infty} \frac{3}{\left(\frac{1}{2}\right)^k}$

C.) $\sum_{k=1}^{\infty} \frac{3}{k^2}$

D.) $\sum_{k=1}^{\infty} \frac{3}{k^{\frac{1}{2}}}$

Strategy

Assume $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n > 0$ for all n .

1. Check if it is a **Geometric Series**.

No! Go to (2).

Yes! If $r \geq 1$ or $r \leq -1$, then the series diverges. If $-1 < r < 1$, then $S = \frac{a_1}{1-r}$.

2. Check if it is a **p-Series**.

No! Go to (3).

Yes! If $p \leq 1$, then the series diverges. If $p > 1$, then the series converges.

3. Check if $\lim_{k \rightarrow \infty} a_k = 0$. (**L'Hôpital's Rule is used if necessary**)

Yes! Then the test is inconclusive. Go to (4).

No! Then the series diverges by the **Divergence Test**.

4. Check if it is a **Telescoping Series**.

No! Go to (5).

Yes! Evaluate S_n by cancelling middle terms (**Partial Fraction Decomposition** is used if necessary) and $S = \lim_{n \rightarrow \infty} S_n$.

5. Use the following Tests:

The Limit Comparison Test / The Comparison Test.

The Ratio Test.

The Integral Test. (when a_n is "easy to integrate")

EXAMPLE: Determine whether the following series converge or diverge using any method.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$$

The Integral Test

Copy Example 4 from the book (pg 722):

You've learned how to prove the divergence of

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

Using the Limit Comparison Test (see https://egunawan.github.io/spring18/notes/hw11_4_to_worksheet_key.pdf).

Follow Example 4 in proving its divergence using the Integral Test.

Extra exam practice questions:

Use any method to determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$

b) $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$

c) $\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$

d) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$