Instruction: Fill in all blanks and examples.

Infinite Series

If we add the terms of a sequence

$$\{a_k\}_{k=1}^n,$$

we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n$$

which is called a (finite) **series** and is also denoted by

$$\sum_{k=1}^{n} a_k.$$

Does it make sense to talk about the sum of infinitely many terms? Consider the **partial sums**

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
S_4 &= a_1 + a_2 + a_3 + a_4,
\end{aligned}$$

and, in general,

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

If the sequence $\{S_n\}_{n=1}^\infty = \{S_1, S_2, S_3, \ldots\}$ of partial sums has limit L, then we say that the infinite series **converges** to L and we write

____

If the sequence $\{S_n\}_{n=1}^n$ of partial sums diverges, then we say that the infinite series **diverges**.

Summary(Notation)

- A **sequence** converges or diverges?

- A **series** converges or diverges?

An important family of infinite series is the geometric series.

Recall

- A **geometric sequence** has the property that each term is obtained by multiplying the previous term by a fixed constant, called the **ratio**, e.g. ______________ .
- Given a geometric sequence $\{a_k\}_{k=1}^{\infty}$, if the ratio is r, then the k-th term can be expressed as $a_k =$ ______________ , e.g. ______________ .
- When ______________ , the sequence converges.

Geometric Series

Partial Sum of Geometric Series

Given a geometric sequence $\{a_k\}_{k=1}^{\infty}$, if the ratio is r, then the sum of the first n terms

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$$

is ______________ .

Why? (Example 2 pg 709)

$$\begin{array}{rccccccccc} S_n = & a_1+ & a_1 r+ & a_1 r^2+ & a_1 r^3+ & \ldots+ & a_1 r^{n-2}+ & a_1 r^{n-1} & \\ r\,S_n = & & a_1 r+ & a_1 r^2+ & a_1 r^3+ & \ldots+ & a_1 r^{n-2}+ & a_1 r^{n-1}+ & a_1 r^n \\ \hline \end{array} -$$

$$\text{Therefore,} \quad S_n - r\,S_n = a_1 - a_1 r^n,$$

$$\text{hence} \quad S_n = a_1 \frac{1 - r^n}{1 - r} \quad \text{if } r \neq 1.$$

Furthermore, since

$$\lim_{n\to\infty} r^n = \begin{cases} \_\_\_\_\_\_ & \text{for } |r| < 1 \\ \_\_\_\_\_\_ & \text{for } r = -1 \\ \_\_\_\_\_\_ & \text{for } |r| > 1 \end{cases}, \quad \text{we have} \quad \lim_{n\to\infty} S_n = \begin{cases} \_\_\_\_\_\_ & \text{for } |r| < 1 \\ \_\_\_\_\_\_ & \text{for } r = -1 \\ \_\_\_\_\_\_ & \text{for } |r| > 1 \end{cases},$$

Theorem (Geometric Series)

Let r and a be real numbers.

If $|r| < 1$, then $\sum_{k=1}^{\infty} ar^{k-1}$ ____________________ .

If $|r| \geq 1$, then $\sum_{k=1}^{\infty} ar^{k-1}$ ____________________ .

Caution

- The **geometric sequence** converges if and only if ____________________ .
- The **geometric series** converges if and only if ____________________ .
- Exercise: If $|r| < 1$, then $\sum_{k=1}^{\infty} ar^{\mathbf{k}}$ ____________________ . Proof: __________ .

  Recall:

  $$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots = \_\_\_\_$$, the area of a 1×1 square.

  $$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots = \_\_\_\_$$, the area of a ____ rectangle.

Example:

Evaluate the geometric series $\sum_{k=1}^{\infty} \frac{7^k}{4^{k+3}}$ or state that it diverges. (At home, use WolframAlpha).

a.) State the test you would use to decide whether the series converges or diverges.
(Hint: so far, we've only learned one test, see above boxed theorem).

b.) (i) Write out the first 4 (four) terms of $\sum_{k=1}^{\infty} \frac{7^k}{4^{k+3}}$.

(ii) Write our the first 4 (four) terms of $\sum_{k=1}^{\infty} ar^{k-1}$.

(iii) Compare terms to find an a and an r so that $\sum_{k=1}^{\infty} \frac{7^k}{4^{k+3}} = \sum_{k=1}^{\infty} ar^{k-1}$.

Example:

Evaluate the geometric series $\sum_{k=2}^{\infty} \frac{2^k}{3^{k-1}}$ or state that it diverges.

(i) Try a slightly different way to solve this problem.

(ii) Come up with a "reality check/sanity check" to convince yourself that your answer makes sense.

Repeating Decimals

Example:

Write $0.9\overline{34} = 0.93434343434\ldots$ as a geometric series and express its value as a fraction.

a.) Can you write $.9\overline{34} = .93434343434\ldots$ as geometric series?

b.) If $0.0\overline{34} = 0.03434343434\cdots = \sum_{k=1}^{\infty} ar^{k-1}$, what is a?

c.) If $0.0\overline{34} = 0.03434343434\cdots = \sum_{k=1}^{\infty} ar^{k-1}$, what is r?

d.) Is $|r| < 1$?

e.) Use the geometric series found in the previous parts to convert $0.9\overline{34} = 0.93434343434\ldots$ into a fraction.

f.) Perform a reality check, for example, verify that your fraction is between $\frac{9}{10}$ and 1.

Example:

Write $2.3\overline{17} = 2.317171717\ldots$ as a geometric series and express its value as a fraction (a ratio of two integers). Compare your work with Example 6, pg 711.

Exercise: observations (First write as a geometric series then convert to a fraction)

- $0.\overline{38}$ $\quad$ $1.2\overline{38}$
- $0.2\overline{74}$ $\quad$ $1.2\overline{74}$

See answer key: `https://egunawan.github.io/spring18/notes/notes11_2part1.pdf`

Answer Key (Do *not* attempt to memorize these observations!)

1.

$$\begin{aligned} 0.\overline{38} &= 0.383838\cdots \\ &= \frac{38}{100} + \frac{38}{100^2} + \frac{38}{100^3} + \cdots \\ &= \frac{\frac{38}{100}}{1 - \frac{1}{100}} \\ &= \frac{38}{99} \end{aligned}$$

Observation: put the repeating digits in the numerator and put N nine's in the denominator, where N is the number of repeating digits.

For example, $0.\overline{123} = \dfrac{123}{999}$

2.

$$\begin{aligned} 1.\overline{38} &= 1 + 0.\overline{38} \\ &= 1 + \frac{38}{99} \\ &= \frac{99+38}{99} \\ &= \frac{(100-1)+38}{99} \\ &= \frac{138-1}{99} \end{aligned}$$

Observation: put all digits minus nonrepeating digits in the numerator and put N nine's in the denominator, where N is the number of repeating digits.

For example, $3.\overline{652} = \dfrac{3652-3}{999}$

3.

$$\begin{aligned}
0.2\overline{74} &= 0.2 + 0.0747474\cdots \\
&= \frac{2}{10} + \frac{74}{1000} + \frac{74}{1000 \times 100} + \frac{74}{1000 \times 100^2} + \cdots \\
&= \frac{2}{10} + \frac{\frac{74}{1000}}{1 - \frac{1}{100}} \\
&= \frac{2}{10} + \frac{74}{990} \\
&= \frac{2 \times 99 + 74}{990} \\
&= \frac{2 \times (100 - 1) + 74}{990} \\
&= \frac{274 - 2}{990}
\end{aligned}$$

Observation: put all digits minus nonrepeating digits in the numerator and put N nine's and M zero's in the denominator, where N is the number of repeating digits and M is the number of nonrepating digits (not including the integer part).

For example, $0.21\overline{95} = \dfrac{2195 - 21}{9900}$

4.

$$\begin{aligned}
1.2\overline{74} &= 1 + 0.2\overline{74} \\
&= 1 + \frac{274 - 2}{990} \\
&= \frac{990 + 274 - 2}{990} \\
&= \frac{(1000 - 10) + 274 - 2}{990} \\
&= \frac{1274 - 12}{990}
\end{aligned}$$

Observation: put all digits minus nonrepeating digits in the numerator and put N nine's and M zero's in the denominator, where N is the number of repeating digits and M is the number of nonrepating digits (not including the integer part).

For example, $9.53\overline{4} = \dfrac{9534 - 953}{900}$