MATH 1152Q ANSWER TO WORKSHEET PROBLEM

Answer to worksheet `https://egunawan.github.io/spring18/notes/notes11_11.pdf` problem 3.

Problem 3:
Use Taylor's inequality to determine a partial sum for the Maclaurin series of $f(x) = \cos x$ that is within 0.0001 of $\cos 2$.

Answer:
Because the center of my Taylor polynomial will be $a = 0$, and I want to approximate $\cos 2$, I can set my interval of approximation to be $|x - 0| \leq 2$.

First, I find a positive number M so that $|f^{(n)}(x)| \leq M$ for all $|x| \leq 2$. A number that works is $\boxed{M = 1}$ because the n-th derivative of $f(x)$ will be either $\pm \sin x$ or $\pm \cos x$.

Next, by Taylor's inequality, we have

$$(\text{error for } T_N(x)) \leq \frac{M}{(N+1)!}|x|^{N+1} = \frac{|x|^{N+1}}{(N+1)!} \text{ for } |x| \leq 2$$

Therefore,

$$(\text{error for } T_N(2)) \leq \frac{|2|^{N+1}}{(N+1)!}$$

So my goal is to find a positive integer N so that

$$\boxed{\frac{2^{N+1}}{(N+1)!} \leq \frac{1}{10^4}}$$

Possible Method 1:
For example, if I choose $N = 5$, then

$$\begin{aligned}\frac{2^6}{6!} &= \frac{\cancel{2}.\cancel{2}.\cancel{2}.2.2.2}{6.5.\cancel{4}.3.\cancel{2}.1} \\ &= \frac{2^3}{6.3} \\ &> \frac{1}{18}, \text{ which is not smaller than } \frac{1}{10^4}, \text{ so we need } N \text{ bigger than 5.}\end{aligned}$$

If I choose $N = 10$, then

$$\begin{aligned}\frac{2^{11}}{11!} &= \frac{2.2.2.2.2.2.2.2.2.2.2}{11.10.9.8.7.6.5.4.3.2.1} \\ &= \frac{2.2.2.2.2.\cancel{2}.\cancel{2}.\cancel{2}.\cancel{2}.\cancel{2}.\cancel{2}}{11.10.9.\cancel{8}.7.6.5.\cancel{4}.3.\cancel{2}.1} \\ &= \frac{2.2.2.2.2}{11.10.9.7.6.5.3} \\ &= \frac{2.2.2}{11.5.9.7.3.5.3} \\ &= \frac{8}{11.(5.7.5).9.(3.3)} \\ &= \frac{8}{11.175.81} \\ &< \frac{1}{10^4}.\end{aligned}$$

So $\boxed{N = 10}$ works, and $T_{10}(x) = 1 - x^2/2! + x^4/4! - x^6/6! + x^8/8! - x^{10}/10!$ would give you an approximation within .0001 for $\cos 2$.

Possible Method 2: I see that

$$\begin{aligned}\frac{|2|^{N+1}}{(N+1)!} &= \frac{2}{(N+1)}\frac{2}{(N)}\frac{2}{(N-1)}\cdots\frac{2}{5}.\frac{2}{4}.\frac{2}{3}.\frac{2}{2}.\frac{2}{1} \\ &= 2\left(\frac{2}{(N+1)}\frac{2}{(N)}\frac{2}{(N-1)}\cdots\frac{2}{5}.\frac{2}{4}.\frac{2}{3}.\frac{2}{2}\right) \\ &< 2\left(\frac{2}{N+1}1.1.\ldots.1.1.1.1\right) \\ &= \frac{4}{N+1}\end{aligned}$$

Since I want

$$\frac{2^{N+1}}{(N+1)!} \leq \frac{1}{10^4},$$

it would be enough if

$$\frac{4}{N+1} \leq \frac{1}{10^4},$$

so it would be enough if

$$4(10^4) \leq N + 1.$$

So $\boxed{N = 40000}$ also works, and $T_40000(x) = 1 - x^2/2! + x^4/4! - x^6/6! + x^8/8! - x^{10}/10! + \cdots + x^{40000}/40000!$ would give you an approximation within .0001 (actually, much smaller) for $\cos 2$.