Taylor's inequality. A bound on the remainder $R_n(x) = f(x) - T_n(x)$, where $T_n(x)$ is the nth-degree Taylor polynomial for $f(x)$ at a, is *Taylor's inequality*:

$$\text{if } |f^{(n+1)}(x)| \leq M \textbf{ for all } |x-a| \leq d \text{ then } \boxed{|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \text{ if } |x-a| \leq d}.$$

Example: Determine the 2nd-degree Taylor polynomial $T_2(x)$ for $\arctan x$ at $a = 1$ and use Taylor's inequality to bound $|R_2(x)|$ if $|x-1| \leq \frac{1}{2}$, where $\arctan x = T_2(x) + R_2(x)$.

Thinking about the problem:

The 2nd-degree Taylor polynomial for a function $f(x)$ at $a = 1$ is

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2.$$

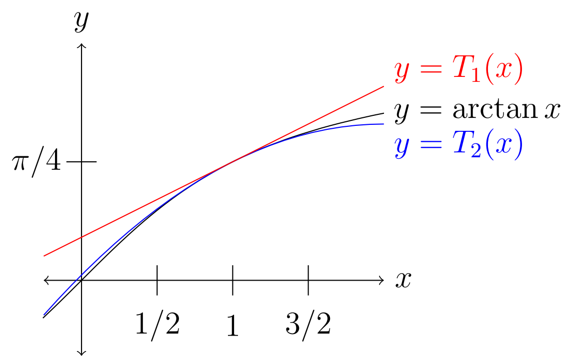
We will find the coefficients when $f(x) = \arctan x$. To bound $R_2(x)$ when $|x-1| \leq \frac{1}{2}$ with Taylor's inequality, we need an M such that $|f'''(x)| \leq M$ for $|x-1| \leq \frac{1}{2}$.

Doing the problem:

To find $T_2(x)$, here is a table of higher derivatives of $f(x) = \arctan x$.

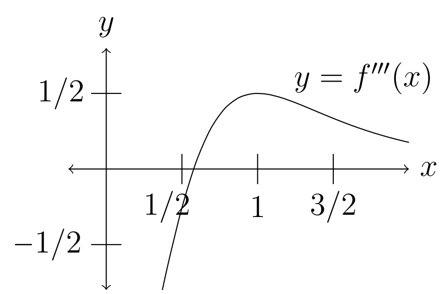
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\arctan x$	$\frac{\pi}{4}$
1	$\frac{1}{1+x^2}$	$\frac{1}{2}$
2	$\frac{-2x}{(1+x^2)^2}$	$-\frac{1}{2}$

From the table, $T_2(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1/2}{2}(x-1)^2 = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2$. The graphs below show $T_2(x)$ is a good approximation of $\arctan x$ for $|x-1| \le \frac{1}{2}$. For comparison we also include $T_1(x)$, the linear approximation to $\arctan x$ at $a = 1$.



To bound $|R_2(x)|$ for $|x-1| \le 1/2$, we need to a number M such that $|f'''(x)| \le M$ for $|x-1| \le 1/2$. What is the biggest value of $|f'''(x)|$ for $|x-1| \le 1/2$?

From the formula for $f''(x)$ in the table, $f'''(x) = \frac{6x^2-2}{(1+x^2)^3}$. Here is the graph of $f'''(x)$.



There is a local maximum of $f'''(x)$ at $x = 1$ where $f'''(1) = 1/2$ (the 4th derivative $f^{(4)}(x) = 24x(1-x^2)/(1+x^2)^4$ vanishes at $x = 1$) and at endpoints $f'''(1/2) \approx -.256$, and $f'''(3/2) \approx .335$, so $-.256 \le f'''(x) \le 1/2$ when $|x-1| \le 1/2$. So use $M = |f'''(1)| = 1/2$:

$$|x-1| \le \frac{1}{2} \Longrightarrow |R_2(x)| \le \frac{M}{3!}|x-1|^3 = \frac{1}{12}|x-1|^3 \le \frac{1}{12}\left(\frac{1}{2}\right)^3 = \frac{1}{12 \cdot 8} = \frac{1}{96} \approx .0104.$$

Solutions should show all of your work, not just a single final answer.

1. **Motivation:** What is the shape of a suspended rope? Images of simple suspended bridges, Finland: https://upload.wikimedia.org/wikipedia/commons/2/24/Soderskar-bridge.jpg. Robert Hooke: https://upload.wikimedia.org/wikipedia/commons/4/48/17_Robert_Hooke_Engineer.JPG. St. Louis arch: https://upload.wikimedia.org/wikipedia/commons/0/00/St_Louis_night_expblend_cropped.jpg.

   For future civil engineers and architects, How the Gateway Arch Got its Shape: YouTube video: https://www.youtube.com/watch?v=vqfVKsBkB1s and article: https://link.springer.com/content/pdf/10.1007/s00004-010-0030-8.pdf

   Determine the 3rd-degree Taylor polynomial $T_3(x)$ for $f(x) = (e^x + e^{-x})/2$ at $a = 0$ and use Taylor's inequality to estimate the error $|f(x) - T_3(x)|$ if $|x| \leq 1$.

   (a) Fill in the following table of higher derivatives for $f(x)$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0		
1		
2		
3		

   (b) Determine the 3rd-degree Taylor polynomial for $f(x)$ at $a = 0$.

   (c) Use Taylor's inequality to bound the error $|f(x) - T_3(x)|$ for $|x| \leq 1$.

2. Use Taylor's inequality to determine an $n > 0$ so that the Taylor polynomial $T_n(x)$ for $\cos x$ at $a = 0$ satisfies $|\cos 2 - T_n(2)| \leq .0001$.

3. The 2nd-degree Taylor polynomial of $\cos x$ at 0 is $1-x^2/2$. Use Taylor's inequality to determine a $d > 0$ for which $|\cos x - (1 - x^2/2)| \leq .001$ for all x in $[-d, d]$.

4. T/F (with justification)

   The 2nd-degree Taylor polynomial for $\sqrt[3]{x}$ at $a = 1$ is $1 + \dfrac{1}{3}(x-1) - \dfrac{2}{9}(x-1)^2$.