**Taylor series and Taylor polynomials of a function at $a$.** If $f(x)$ can be written as a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ in an interval around $a$ then $c_n$ must be $\dfrac{f^{(n)}(a)}{n!}$. We call

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

the *Taylor series* of $f(x)$ at $a$ and we call the partial sum

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the *nth-degree Taylor polynomial* of $f(x)$ at $a$. Ideally $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$ for $x$ near $a$, or equivalently $f(x) = \lim_{n\to\infty} T_n(x)$ for $x$ near $a$, and to verify this in examples we can use Taylor's inequality below.

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**Maclaurin series.** The Taylor series of $f(x)$ at $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

and is called the *Maclaurin series* of $f(x)$[1]. Ideally $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!}x^n$ for $x$ near 0.

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**Taylor's inequality**. A bound on the remainder $R_n(x) = f(x) - T_n(x)$, where $T_n(x)$ is a Taylor polynomial for $f(x)$ at $a$, is *Taylor's inequality*, which uses a bound on $|f^{(n+1)}(x)|$:

$$\text{if } \boxed{|f^{(n+1)}(x)| \leq M \textbf{ for all } |x-a| \leq d}, \text{ then } \boxed{|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \text{ if } |x-a| \leq d}.$$

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**Important Maclaurin series representations.**

| Function | Validity | Function | Validity |
|---|---|---|---|
| $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ | $-1 < x < 1$ | $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$ | all $x$ |
| $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ | all $x$ | $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$ | all $x$ |
| $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}$ | $-1 < x \leq 1$ | $\arctan x = \sum_{n=1}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ | $-1 \leq x \leq 1$ |

[1]The term "Maclaurin series" has a peculiar status: it essentially exists only in calculus courses. People who use power series regularly, in math or physics, speak instead about a Taylor series or power series at 0.

**Example:** (1.) Compute the Taylor series for $f(x) = \ln(x)$ at $a = 10$, and (2.) use Taylor's inequality to show when $|x - 10| \leq 4$ that $|R_n(x)| = |\ln(x) - T_n(x)| \to 0$ as $n \to \infty$.

*Think:* Differentiate $\ln x$ enough times to see a pattern. The pattern will give us the coefficients in the Taylor series and help us bound $|f^{(n+1)}(x)|$ to find $M$ in Taylor's inequality.

*Doing the problem:* The first several higher derivatives of $f(x) = \ln x$ are in the table below.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $f^{(n)}(x)$ | $\ln x$ | $1/x$ | $-1/x^2$ | $2/x^3$ | $-6/x^4$ | $24/x^5$ | $-120/x^6$ | $720/x^7$ |

The pattern for $n \geq 1$ is $f^{(n)}(x) = (-1)^{n-1}\dfrac{(n-1)!}{x^n}$, so the Taylor series of $\ln x$ at $a = 10$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(10)}{n!}(x-10)^n &= f(10) + \sum_{n=1}^{\infty} \frac{f^{(n)}(10)}{n!}(x-10)^n \\
&= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{10^n n!}(x-10)^n \\
&= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-10)^n}{10^n n} \\
&= \ln 10 + \frac{x-10}{10} - \frac{(x-10)^2}{200} + \frac{(x-10)^3}{3000} - \frac{(x-10)^4}{40000} + \cdots
\end{aligned}$$

Now find an $M$ so that $|f^{(n+1)}(x)| \leq M$ when $|x - 10| \leq 4$, which means $6 \leq x \leq 14$.

Since $f^{(n+1)}(x) = (-1)^n \dfrac{n!}{x^{n+1}}$, we need $\left|\dfrac{n!}{x^{n+1}}\right| \leq M$ for $6 \leq x \leq 14$. The biggest value of $\left|\dfrac{n!}{x^{n+1}}\right| = \dfrac{n!}{x^{n+1}}$ in that $x$-range is $\dfrac{n!}{6^{n+1}}$, so use $\boxed{M = \dfrac{n!}{6^{n+1}}}$: if $|x - 10| \leq 4$ then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-10|^{n+1} = \frac{n!/6^{n+1}}{(n+1)!}|x-10|^{n+1} = \frac{1}{n+1}\left(\frac{|x-10|}{6}\right)^{n+1} \leq \frac{(2/3)^{n+1}}{n+1}.$$

Thus $R_n(x) \to 0$ as $n \to \infty$, so for $|x - 10| \leq 4$, $\ln x$ equals its Taylor series at $a = 10$.

**Solutions should show all of your work, not just a single final answer.**

1. Let $f(x) = \sqrt{x}$.

   (a) Does $f(x)$ have a Maclaurin series? Why or why not?

   (b) Determine the 3rd-degree Taylor polynomial $T_3(x)$ for $f(x) = \sqrt{x}$ at $a = 9$. Start off by filling in the following table of higher derivatives for $f(x)$.

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(9)$ |
|---|---|---|
| 0 | | |
| 1 | | |
| 2 | | |
| 3 | | |

   (c) Compute $T_3(10)$ from (b). (This is an estimate for $\sqrt{10}$.)

   (d) Use Taylor's inequality to bound the error $|\sqrt{10} - T_3(10)|$.

(e) Use a computing tool to confirm that the error is smaller than the error bound you stated in part (d).

2. Use the Maclaurin series for $e^x$ and $\arctan x$ to find the Maclaurin series for the following functions. Determine the radius of convergence in each case.

   (a) $f(x) = e^{3x} + e^{-3x}$

   (b) $f(x) = \arctan\left(\frac{x}{3}\right)$

3. T/F (with justification)

   If $f(x) = 1 + 3x - 2x^2 + 5x^3 + \cdots$ for $|x| < 1$ then $f'''(0) = 30$.