Recall (fill in the blank, see middle of pg 761)

The n-th Taylor Polynomial centered at a is

____.

Remainder in a Taylor Polynomial

Taylor polynomials provide good approximations to functions near a specific point, but how good are the approximations?

Let $R_n(x) = f(x) - T_n(x)$, then $R_n(x)$ is called the remainder of the Taylor series.

(Copy from pg 762) Theorem Taylor's Inequality

Suppose there exists a number M such that

$$\left|f^{(n+1)}(x)\right| \le M \text{ for } |x-a| \le d,$$

then the remainder $R_n(x)$ of the Taylor series satisfies

$|R_n(x)| \le$ ________________________ for ________________________ .

Follow Sec 11.11 (next section) Example 1 pg 775-776: Consider the function $f(x) = \sqrt[3]{x}$.

a. Find the **Taylor polynomials of order 2** centered at $x = 8$ for $f(x)$.

b. How accurate is this approximation when $7 \le x \le 9$?

Using Taylor series to solve other problems.

1. The Maclaurin series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $-\infty < x < \infty$.

   Evaluate $\lim_{x \to 0} \frac{e^x - 1}{x}$ using Maclaurin series.

   Evaluate $\lim_{x \to 0} \frac{e^x - 1}{x}$ using L'hospital rule.

   Verify your result $\lim_{x \to 0} \frac{e^x - 1}{x}$ using a graphing tool if you have one.

2. The Maclaurin series for $\sin x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for $-\infty < x < \infty$.

   Assume the conditions for the **Integral Test** have been verified. Determine the convergence or divergence of the series $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$.

[Solution of Problem 2]

Let $f(x)=\sin\left(\frac{1}{x}\right)$ for $x\geq 1$.

Note that $0<\frac{1}{x}\leq 1<\frac{\pi}{2}$, so $\frac{1}{x}$ is in the first quadrant.
Hence $f(x)$ is positive for $x\geq 1$.
In addition, $f(x)$ is continuous for $x\geq 1$.
Furthermore, $f'(x)=-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)<0$ for $x\geq 1$.
Thus $f(x)$ is decreasing for $x\geq 1$.

Therefore, the Integral Test applies.

For $x\geq 1$, $\sin\left(\frac{1}{x}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{x}\right)^{2n+1}$

$$=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{-2n-1}$$

Thus $\int\sin\left(\frac{1}{x}\right)dx=\int\left[\frac{1}{x}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{-2n-1}\right]dx$

$$=\ln|x|-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n(2n+1)!}x^{-2n}+C$$

Therefore, $\int_1^{\infty}\sin\left(\frac{1}{x}\right)dx=\lim_{t\to\infty}\int_1^t\sin\left(\frac{1}{x}\right)dx$

$$=\lim_{t\to\infty}\left[\ln|x|-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n(2n+1)!}x^{-2n}\right]_1^t$$

$$=\lim_{t\to\infty}\left[\ln|t|-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n(2n+1)!}\frac{1}{t^{2n}}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n(2n+1)!}\right]$$

Note that $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n(2n+1)!}$ is absolutely convergent by the Ratio Test since

$$\lim_{x\to\infty}\frac{\frac{1}{(2n+2)(2n+3)!}}{\frac{1}{2n(2n+1)!}}=\lim_{x\to\infty}\frac{2n(2n+1)!}{(2n+2)(2n+3)!}=\lim_{x\to\infty}\frac{n}{(n+1)(2n+3)(2n+2)}=0$$

As a result, $\int_1^{\infty}\sin\left(\frac{1}{x}\right)dx=\lim_{t\to\infty}\left[\ln|t|-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n(2n+1)!}\frac{1}{t^{2n}}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n(2n+1)!}\right]$

$$=\infty$$

The improper integral $\int_1^{\infty}\sin\left(\frac{1}{x}\right)dx$ diverges.

Therefore, the infinite series $\sum_{k=1}^{\infty}\sin\left(\frac{1}{k}\right)$ diverges by the Integral Test.